You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. In your solution you can use only statements that were proven in class.

There are 3 questions, each divided into several subquestions. Work on as many questions or subquestions as you wish. The complete solution of one question will be considered more that two half solutions. A total of 100 points will be considred a very good score.

Name: ____________________

Question:	1	2	3	Total
Points:	35	65	100	200
Score:				

1. (5 points) Consider the partial differential equation

$$u_t(t, x_1, x_2) + x_1 u_{x_1}(t, x_1, x_2) + x_2 u_{x_2}(t, x_1, x_2) = 0 \tag{1}$$

where $u_t(t, x_1, x_2) = \partial_t u(t, x_1, x_2)$ and $u_{x_i}(t, x_1, x_2) = \partial_{x_i} u(t, x_1, x_2)$.

(a) (10 points) Let $u(t, x_1, x_2)$ be a solution of eq. (1). Define

$$g(s) = u\left(t - s, e^{-s}x_1, e^{-s}x_2\right).$$

Show that

$$g'(s) = 0$$

where $' = \frac{d}{ds}$.

Solution: Call $(y_1, y_2) = e^{-s}(x_1, x_2)$ We have

$$g'(s) = -u_t(t - s, y_1, y_2) - y_1 \partial_{y_1} u(t - s, y_1, y_2) - y_2 \partial_{y_2} u(t - s, y_1, y_2) = 0.$$

(b) (10 points) Use subquestion a) to find a solution of eq.(1) with initial conditions:

$$u(0, x_1, x_2) = u_0(x_1, x_2).$$

Solution: It is enough to observe that

$$g(0) = u(t, x_1, x_2) \qquad g(t) = u(0, e^{-t}x_1, e^{-t}x_2) = u_0(e^{-t}x_1, e^{-t}x_2).$$

Since $g(0) = g(t)$ we get

$$u(t, x_1, x_2) = u_0(e^{-t}x_1, e^{-t}x_2).$$

(c) (10 points) Extend subquestions a) and b) to find a solution of

$$\begin{aligned} &u_t(t,x_1,x_2) + x_1 u_{x_1}(t,x_1,x_2) + x_2 u_{x_2}(t,x_1,x_2) = f(t,x_1,x_2) \\ &u(0,x_1,x_2) = u_0(x_1,x_2) \end{aligned} \tag{2}$$

where $f \in C^0(\mathbb{R} \times \mathbb{R}^2)$.

Solution: Reasoning like in subquestion a) we get

$$g'(s) = -\tilde{f}(s)$$

where

$$\tilde{f}(s) = f(t-s, e^{-s}x_1, e^{-s}x_2).$$

We thus have

$$g(t) - g(0) = \int_0^t \tilde{f}(s)ds$$

or

$$\begin{aligned} u(t,x_1,x_2) =& u_0(e^{-t}x_1, e^{-t}x_2) + \int_0^t f(t-s, e^{-s}x_1, e^{-s}x_2)ds \\ =& u_0(e^{-t}x_1, e^{-t}x_2) + \int_0^t f(s, e^{t-s}x_1, e^{t-s}x_2)ds. \end{aligned}$$

2. (10 points) Let $u \in C^2(\mathbb{R} \times \mathbb{R}^n)$ be a solution of

$$u_{tt}(t,x) - \Delta u(t,x) = 0.$$

(a) (15 points) Define

$$v(t,x) = u(t, Ox)$$

where O is a $n \times n$ unitary matrix, that is $O^* O = \mathrm{Id}$. Show that v satisfies

$$v_{tt}(t,x) - \Delta v(t,x) = 0.$$

(**Hint**: call $y = Ox$ and compute $\partial_{x_i} v(x)$ in term of $\partial_{y_j} u(y)$. Proceed similarly for $\partial_{x_i}^2 v(x)$.)

Solution: Calling $y = Ox$ we get

$$\partial_{x_i} v(t,x) = \sum_j O_{i,j} \partial_{y_j} u(t,y)$$

and

$$\partial_{x_i}^2 v(t,x) = \sum_j \sum_k O_{i,j} O_{i,k} \partial_{y_k} \partial_{y_j} u(t,y)$$

thus

$$\Delta_x v(t,x) = \sum_i \partial_{x_i}^2 v(t,x) = \sum_{j,k} \left(\sum_i O_{i,j} O_{i,k} \right) \partial_{y_k} \partial_{y_j} u(t,y).$$

but

$$\sum_i O_{i,j} O_{i,k} = (O^* O)_{j,k} = \delta_{j,k}$$

so that

$$\Delta_x v(t,x) = \Delta_y u(t,y).$$

Since

$$v_{tt}(t,x) = \partial_t^2 u(t,y)$$

so that

$$v_{tt}(t,x) - \Delta_x v(t,x) = \partial_t^2 u(t,y) - \Delta_y u(t,y) = 0.$$

(b) (15 points) (**No Galileian Invariance**) Given $g \in \mathbb{R}^n$, set

$$G(t,x) = (t, x - gt)$$

and

$$v^G(t,x) = u\left(G(t,x)\right).$$

Show that in general

$$v^G_{tt}(t,x) - \Delta v^G(t,x) \neq 0.$$

(**Hint**: start with $n = 1$.)

Solution: Let $h_0(x_1)$ be in $C^2(\mathbb{R})$. It is easy to heck that

$$u(t,x) = h_0(x_1 + t)$$

solves the wave equation. Take now $g = (1, 0, \ldots, 0)$ so that

$$v^G(t,x) = h_0(x_1)$$

Clearly $v^G(t,x)$ does not solve the wave equation as soon as h_0 is not a constant.

(c) (25 points) (**Relativistic Invariance**) Given $g \in \mathbb{R}^n$ with $|g| < 1$, set

$$L(t,x) = (\gamma(t - g \cdot x), x - (1-\gamma)(x \cdot \hat{g})\hat{g} - \gamma g t)$$

where

$$\gamma = \frac{1}{\sqrt{1-|g|^2}} \qquad \text{and} \qquad \hat{g} = \frac{g}{|g|}$$

and $\cdot$ is the usual scalar product in $\mathbb{R}^n$. Define

$$v^L(t,x) = u\left(L(t,x)\right).$$

Show that

$$v^L_{tt}(t,x) - \Delta v^L(t,x) = 0.$$

(**Hint**: Use subquestion a) to bring g to a "simple" vector. Call $t = x_0$ and $\bar{x} = (x_0, x)$ to simplify notations. Observe that $L(t,x)$ can be written as $L\bar{x}$ for a suitable matrix L. Proceed as in a).)

Solution:

We first assume that $g = (g_1, 0, \ldots, 0)$. For such a g we get

$$L(t,x) = (\gamma(t - g_1 x_1), \gamma(x_1 - g_1 t), x_2, \ldots, x_n)$$

Lets call $t = x_0$ and $\bar{x} = (x_0, x)$.Observe that $L(\bar{x})$ is a linear transforamtion, that is $L(\bar{x}) = L\bar{x}$ where

$$L = \begin{pmatrix} \gamma & -\gamma g_1 & 0_{1\times n-1} \\ -\gamma g_1 & \gamma & 0_{1\times n-1} \\ 0_{n-1\times 1} & 0_{n-2\times 1} & \mathrm{Id_{n-1}} \end{pmatrix}$$

where $\mathrm{Id_n}$ is the $n \times n$ identity matrix and $0_{n\times m}$ is the $n \times m$ matrix of all 0. Like in subquestion a) we set $\bar{y} = L\bar{x}$ and we get

$$\partial_{x_i} v^L(\bar{x}) = \sum_j L_{i,j} \partial_{y_j} u(\bar{y})$$

and

$$\partial^2_{x_i} v^L(\bar{x}) = \sum_j \sum_k L_{i,j} L_{i,k} \partial_{y_j} \partial_{y_k} u(\bar{y})$$

so that

$$\partial^2_{x_0} v^L(\bar{x}) - \sum_i \partial^2_{x_i} v^L(\bar{x}) = \sum_j \sum_k \left(\sum_i \sigma_i L_{i,j} L_{i,k} \right) \partial_{y_j} \partial_{y_k} u(\bar{y})$$

where $\sigma_0 = 1$ while $\sigma_i = -1$ for $i \geq 1$. It is now easy to chck that

$$\sum_i \sigma_i L_{i,j} L_{i,k} = \sigma_j \delta_{j,k}$$

so that

$$v^L_{tt}(t,x) - \Delta_x v^L(t,x) = \partial^2_{x_0} v^L(\bar{x}) - \sum_i \partial^2_{x_i} v^L(\bar{x}) = \partial^2_{y_0} u(\bar{y}) - \sum_i \partial^2_{y_i} u(\bar{y}) = 0.$$

Observe now that, if $\mathcal{O}$ is the $(n+1)\times(n+1)$ matrix of the form

$$\mathcal{O} = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & O \end{pmatrix}$$

where O is a unitary $n \times n$ matrix, then

$$\mathcal{O} L_g \bar{x} = L_{Og} \mathcal{O} \bar{x}$$

Chose O such that Og is parallel to the x_1 axis. Because $\mathcal{O}$ is unitary we get

$$u(L_g \bar{x}) = u(\mathcal{O}^* L_{Og} \mathcal{O} \bar{x}).$$

The fact that u is a solution implyes that $u(\mathcal{O}\bar{x})$ is a solution (from a)). Thus $u(L_{Og}\mathcal{O}\bar{x})$ is a solution (from the above argument). Finally, again from a), $u(\mathcal{O}^* L_{Og} \mathcal{O} \bar{x})$.

3. (10 points) Let U be an open set in $\mathbb{R}^n$, $n \geq 3$, $x_0 \in U$ and u and harmonic function on $U \backslash \{x_0\}$, that is
$$\Delta u(x) = 0 \qquad\qquad x \in U,\ x \neq x_0.$$
Assume moreover that u is bounded on $U \backslash \{x_0\}$, that is
$$M = \sup_{x \in U \backslash \{x_0\}} |u(x)| < \infty.$$
In the following we will show that u can be extended to an harmonic function on U.

(a) (15 points) Let $x \in U$ and $r, s > 0$ be such that $B(s, x_0) \subset B(r, x) \subset U$. Show that
$$\frac{1}{r}\int_{\partial B(r,x)} Du(y) \cdot (y - x) dS(y) - \frac{1}{s}\int_{\partial B(s,x_0)} Du(y) \cdot (y - x_0) dS(y) = 0.$$
(**Hint**: look at the proof of the mean value formula.)

Solution: Observe that
$$\frac{1}{r}\int_{\partial B(r,x)} Du(y) \cdot (y - x) dS(y) - \frac{1}{s}\int_{\partial B(s,x_0)} Du(y) \cdot (y - x_0) dS(y) =$$
$$\int_{\partial B(r,x)} \partial_\nu u(y) dS(y) - \int_{\partial B(s,x_0)} \partial_\nu u(y) dS(y) =$$
$$\int_{\partial A} \partial_\nu u(y) dS(y)$$
where $A = B(r, x) \backslash B(s, x_0)$. Clealry $x_0 \notin A$ so that, using Green's formula, we get
$$\int_{\partial A} \partial_\nu u(y) dS(y) = \int_A \Delta u(y) dy = 0.$$

(b) (15 points) Let

$$\psi(s) = \int_{B(s,x_0)} u(y)dy.$$

Show that

$$s^{n-1}\left(\frac{1}{s^{n-1}}\psi'(s)\right)' = \frac{1}{s}\int_{\partial B(s,x_0)} Du(y)\cdot(y-x_0)dS(y).$$

where $' = \frac{d}{ds}$. (**Hint**: you can refer to the book for most of the computations.)

Solution: We have

$$\frac{d}{ds}\psi(s) = \int_{\partial B(s,x_0)} u(y)dS(y)$$

so that

$$\frac{1}{n\alpha(n)s^{n-1}}\frac{d}{ds}\psi(s) = ⨏_{\partial B(s,x_0)} u(y)dS(y) = \phi(s)$$

From the book we know that

$$\frac{d}{ds}\phi(s) = \frac{1}{s}⨏_{\partial B(s,x_0)} Du(y)\cdot(y-x_0)dS(y).$$

Combining we get the thesis.

(c) (20 points) Use subquestion a) and b) to show that

$$\frac{1}{s}\int_{\partial B(s,x_0)} Du(y)\cdot(y-x_0)dS(y)=0.$$

for every s. (**Hint**: use a) to solve the equation in subquestion b) and compare the solution with the estimate of $\psi(s)$ you get from boundedness.)

Solution: Point a) tell us that

$$\frac{1}{s}\int_{\partial B(s,x_0)} Du(y)\cdot(x_0-y)dS(y)=C.$$

From subquestion b) we get

$$s^{n-1}\left(\frac{1}{s^{n-1}}\psi'(s)\right)'=C$$

or

$$\psi(s)=-\frac{Cs^2}{2(n-2)}+\frac{C_1 s^n}{n}+C_2$$

for suitable contants C_1 and C_2. On the other hand, because $u<M$ we know that

$$|\psi(s)|\le Ms^n.$$

This immediately implyes that $C_2=0$. Moreover we must have

$$\left|-\frac{Cs^2}{2(n-2)}+\frac{C_1 s^n}{n}\right|\le Ms^n$$

or

$$\left|-\frac{C}{2(n-2)s^{n-2}}+\frac{C_1}{n}\right|\le M$$

and this is possible only if $C=0$.

(d) (15 points) Let

$$\phi(r,x) = \fint_{\partial B(r,x)} u(y)dS(y).$$

Show that, for every $x \in U$, $\phi(r,x)$ does not depend on r. (**Hint**: just compute $\phi'(r,x)$ and use a), b) and c).)

Solution: As usual

$$\phi'(r,x) = \frac{1}{r}\fint_{\partial B(s,x_0)} Du(y) \cdot (y - x_0)dS(y).$$

If $x = x_0$ the statement follows immediately from c). If $r < |x - x_0|$ it is a consequence of the mean value formula while if $r > |x-x_0|$ it follows immediately from a) and c).

It remain to check that $\phi(r,x)$ is continuous when $r = |x - x_0|$ but this follows easily from the boundedness of u.

(e) (25 points) Show that $u(x)$ can be extended to a continuous function on U. Show that the extension of u is $C^2(U)$. Complete the proof by showing that it is harmonic. (**Hint**: write $u(x)$ and $u(x')$ as integrals on suitable balls. Compare the integral when x is close to x'. Repeat for $\partial_{x_i} u$ and ...)

Solution: Define $u(x_0) = \phi(r, x_0)$. Observe that we have, for every $x \in U$,

$$\fint_{B(r,x)} u(y)dy = u(x)$$

Let x_n a sequence of subquestion such that $x_n \to x_0$. We have

$$\begin{aligned} |u(x_n) - u(x_0)| = &\left| \fint_{B(r,x)} u(y)dy - \fint_{B(r,x_n)} u(y)dy \right| \leq \\ \leq &\frac{1}{n\alpha(n)r^n} \int_{B(r,x)\Delta B(r,x_n)} |u(y)|dy \longrightarrow_{n\to\infty} 0 \end{aligned}$$

Thus u is a continuous function.
Observe now that $\partial_{x_i} u$ is an harmonic function on $U \backslash \{x_0\}$. Moreover for every $|x - x_0| \leq \epsilon$ and $r > 2\epsilon$ we have

$$\partial_{x_i} u(x) = \fint_{\partial B(0,1)} \partial_{x_i} u(x + ry)dS(y) = \fint_{\partial B(x,r)} \partial_{x_i} u(y)dS(y)$$

Since $B(x, r) \subset B(x_0, 2r)$ and $\partial_{x_i} u$ is continuos on $U \backslash B(x_0, 2r)$ and we get that $\partial_{x_i} u$ is continuous at x_0 Similarly we get that $\partial_{x_i}\partial_{x_j} u$ is continuous at x_0 so that $u \in C^2(U)$. By continuity we have that

$$\Delta u(x_0) = 0$$

so that u is harmonic on U.