1.5 n 4 (3)

We can reason by induction on n.

If $n=1$ we ~~have~~ need To show

$$\frac{d^m}{dx^m}(u v) = \sum_{0\le k\le m} \frac{d^k}{dx^k} u \, \frac{d^{m-k}}{dx^{m-k}} v \binom{m}{k}$$

Again by induction: Evident for $m=1$

$$\frac{d^{m+1}}{dx^m}(u v) = \frac{d}{dx}\left(\frac{d^m}{dx^m}(u \cdot v)\right) =$$

$$\sum_{0\le k\le m}\binom{m}{k}\left(\frac{d^{k+1}}{dx^{k+1}} u \, \frac{d^{m-k}}{dx^{m-k}} + \frac{d^k}{dx^k} u \, \frac{d^{m-k+1}}{dx^{m-k+1}} v\right) =$$

$$\sum_{0\le k\le m+1}\left[\binom{m}{k-1} + \binom{m}{k}\right]\frac{d^k}{dx^k} u \, \frac{d^{m+1-k}}{dx^{m+1-k}} v$$

where $\binom{m}{-1} = \binom{m}{m+1} = 0$. Since

$\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}$ we get

the Thesis.

If now $u, v : \mathbb{R}^{n+1} \to \mathbb{R}$ we have

$$D^{\alpha}(u \cdot v) = \partial^{n}_{x_{n+1}} D^{\bar{\alpha}}(u v)$$

where $\bar{\alpha}_i := \alpha_i$ if $i = 1 \dots n$ and $\bar{\alpha}_{n+1} = 0$. The thesis easily follows applying the formula to $D^{\bar{\alpha}}(uv)$ looking at u, v as function $: \mathbb{R}^n \to \mathbb{R}$ with x_{n+1} fixed.

1.5 n B(4)

We start from Taylor expansion ~~m~~ for

$n=1$:

$$f(x)=\sum_{n=0}^{K} f^{(k)}(0)\frac{x^{k}}{k!}+f^{(k+1)}(\bar{x})\frac{x^{k+1}}{(k+1)!}$$

for some point $\bar{x}\in(0,x)$.

Let now, for $f:\mathbb{R}^n\to\mathbb{R}$,

$$g(t)=f(tx)$$

so that

$$g(1)=\sum_{n=0}^{K} g^{(n)}(0)\frac{1}{n!}+g^{(k+1)}(\bar{t})\frac{1}{(k+1)!} \qquad (1)$$

Observe that

$$g(0)=f(0)$$

$$g^{(1)}(0)=\frac{d}{dt}f(tx_1,tx_2,\dots tx_n)\Big|_{t=0}=$$

$$x_1\partial_{x_1}f(\bar{t}x)+x_2\partial_{x_2}f(\bar{t}x)+\dots+x_n\partial_{x_n}f(\bar{t}x)$$

$$=\sum_{|\alpha|=1}x^{\alpha}D^{\alpha}f(\bar{t}x)$$

Assume That
$$g^{(n-1)}(\bar{t}) = \sum_{|\alpha|=n-1} \frac{(n-1)!}{\alpha!} D^\alpha f(\bar{t}x) x^\alpha$$

Inductively

$$g^{(n)}(\bar{t}) = \frac{d^{n-1}}{dt^{n-1}}\left(\frac{d}{dt} g(t)\right)\Big|_{t=\bar{t}} =$$

$$\frac{d^{n-1}}{dt^{n-1}}\left(\sum_j x_j \partial_{x_j} f(tx)\right)_{t=\bar{t}} =$$

$$\sum_j x_j \frac{d^{n-1}}{dt^{n-1}} \partial_{x_j} f(\bar{t}x) =$$

$$\sum_j x_j \sum_{|\alpha|=n-1} D^\alpha \partial_{x_j} f(\bar{t}x) x^\alpha \frac{(n-1)!}{\alpha!}$$

$$\sum_{|\alpha|=n}\left(\sum_{j=1}^{n} \frac{(n-1)!}{\alpha_1!\alpha_2!\dots(\alpha_j-1)!\dots\alpha_n!}\right) D^\alpha f(\bar{t}x) x^\alpha$$

But as before

$$\sum_{j=1}^{n} \frac{(n-1)!}{\alpha_1!\alpha_2!\dots(\alpha_j-1)!\dots\alpha_n!} = \frac{n!}{\alpha!}$$

so That

$$g^{(n)}(\bar{t}) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} D^\alpha f(\bar{t}x) x^\alpha$$

Substituting in (1) we get

$$f(x) = \sum_{n=0}^{k} \sum_{|\alpha|=n} D^{\alpha} f(0) \frac{x^{\alpha}}{\alpha!} + \sum_{|\alpha|=k+1} D^{\alpha} f(\bar{t}x) \frac{x^{\alpha}}{\alpha!}$$

Observe now that

$$x_i \leq \cancel{\sqrt{}}\, |x|$$

so that

$$x^{\alpha} \leq |x|^{|\alpha|}$$

and

$$\frac{1}{|x|^{k+1}} \sum_{|\alpha|=k+1} D^{\alpha} f(\bar{t}x) \frac{x^{\alpha}}{\alpha!} \leq$$

$$\sum_{|\alpha|=k+1} \left| D^{\alpha} f(\bar{t}x) \right| \frac{1}{\alpha!} < +\infty$$

so that

$$\sum_{|\alpha|=k+1} D^{\alpha} f(\bar{t}x) \frac{x^{\alpha}}{\alpha!} = O(|x|^{k+1})$$

2.5 n.1.

Let $u(x,t)$ be a solution. Consider

$$g(s) = u(x - sb, t - s)$$

Then we have

$$\dot{w}(s) = -b\, Du(x-sb, t-s) - u_t(x-sb, t-s) = c\, u(x-sb, t-s)$$

so that we get

$$\dot{w}(s) = c\, w(s)$$

That is

$$w(s) = w(0)\, e^{cs}$$

or

$$u(x,t) = w(0) = e^{-ct} w(t) = e^{-ct} u(x - tb, 0) = g(x - tb)\, e^{-ct}$$

Thus

$$u(x,t) = e^{-ct} g(x - tb)$$

solves the initial value problem.

2.5 n 4

(a) Like in the proof of Th. 2 we set

$$\phi(r) = ⨍_{\partial B(x,r)} u(y)\, dS(y)$$

We get

$$\phi'(r) = \frac{r}{n} \int_{B(x,r)} \Delta u(y)\, dy \geq 0$$

Since again $\phi(0) = u(x)$ we get

$$u(x) \leq ⨍_{\partial B(x,r)} u(y)\, dS(y)$$

Again

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \left(\int_{\partial B(x,s)} u\, dS \right) ds \geq$$

$$\geq u(x) \int_0^r n\alpha(n) s^{n-1}\, ds = \alpha(n) r^n u(x)$$

(b) Assume there is $\bar{x}$ in U such that

$u(\bar{x}) = \max_{x \in \bar{U}} u(x)$. Then if $B(\bar{x}, r) \subset U$

we have

$$u(\bar{x}) \geq \fint_{B(\bar{x},r)} u(y)\,dy$$

but also

$$u(\bar{x}) \leq \fint_{B(x,r)} u(y)\,dy$$

This implies that

$$u(y) = u(\bar{x}) \text{ for } y \in B(\bar{x}, r)$$

As in the proof of Th. 4 we thus have

that

$$u(y) = u(\bar{x}) \text{ for } y \text{ in the}$$

connected component of U containing $\bar{x}$.

This easily implies (6b).

(c) Observe That

$$\partial_{x_i}^2 \phi(u) = \phi''(x)\left(\partial_{x_i} u\right)^2 + \phi'(u)\, \partial_{x_i}^2 u$$

so that

$$\Delta \phi(u) = \phi''(x) \sum_{i=1}^{n} \left(\partial_{x_i} u\right)^2 + \phi'(u)\, \Delta u$$

since u is harmonic $\Delta u = 0$
since ϕ is convex $\phi''(x) \geq 0$
clearly

$$\sum_{i=1}^{n} \left(\partial_{x_i} u\right)^2 \geq 0$$

so That

$$\Delta \phi(u) \geq 0$$

(d) Observe that

$$v := \sum_{i=1}^{n} \left(\partial_{x_i} u \right)^2$$

We know that $\partial_{x_i} u$ is harmonic, and $\phi(x) = x^2$ is convex so that

$$\Delta \left(\partial_{x_i} u \right)^2 \geq 0 \quad \text{by point (c).}$$

Finally

$$\Delta v = \sum_{i=1}^{n} \Delta \left(\partial_{x_i} u \right)^2 \geq 0$$

2.5. 4 (3)

Like in The proof of Th. 2 leT

$$\phi(s) = \fint_{\partial B(0,s)} u(y)\, dS(y)$$

Again we get

$$\phi'(s) = \frac{s}{n} \fint_{B(0,s)} \Delta u(y)\, dy$$

We have

$$\phi(r) - \phi(0) = \int_0^r \phi'(s)\, ds \qquad (1)$$

Clearly

$$\phi(0) = u(0)$$

while

$$\phi(r) = \int_{\partial B(0,r)} u(y)\, dy = \int_{\partial B(0,r)} g(y)\, dy$$

$$\int_0^r \phi'(s)\,ds = -\int_0^r ds \frac{1}{n\alpha(n)s^{n-1}} \int_0^s dt \int_{\partial B(0,t)} f(y)\,dS(y) =$$

$$= -\int_0^r dt \int_t^r ds \frac{1}{n\alpha(n)s^{n-1}} \int_{\partial B(0,t)} f(y)\,dS(y) =$$

$$= \frac{+1}{n\alpha(n)} \int_0^r dt \frac{1}{n-2}\left(\frac{1}{r^{n-2}} - \frac{1}{t^{n-2}}\right) \int_{\partial B(0,t)} f(y)\,dS(y) =$$

$$= +\frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{r^{n-2}} - \frac{1}{|x|^{n-2}}\right) f(x)\,dx$$

Substituting in (1) we get the Thesis.