n 5.

Let $G(x,y)$ be The Green function for $B(0,1)$. We observe That

a) $G(x,y) \geq 0$ for $x, y \in B(0,1)$. Indeed fixed x $G(x,y)$ is harmonic ~~on~~ y on

$$B(0,1) \setminus B(x,\varepsilon)$$

for every ε. Moreover $G(x,y)=0$ for $y \in \partial B(0,1)$ and $G(x,y)>0$ ~~on~~ for $y \in \partial B(x,\varepsilon)$ for ε small enough.

b) $\partial_\nu G(x,y) = K(x,y) \geq 0$ (from explict formula).

Thus

$$|u(x)| \leq \left|\int_{B(0,1)} G(x,y) f(y)\,dy\right| + \left|\int_{\partial B(0,1)} \frac{\partial}{\partial\nu} G(x,y)\, g(y)\,dS_y\right|$$

$$\leq \sup_{B(0,1)} |f(y)| \int_{B(0,1)} G(x,y)\,dy + \sup_{\partial B(0,1)} g(y) \int \frac{\partial}{\partial\nu} G(x,y)\,dS(y)$$

By The Theory we have That

$$h(x) = +\int_{B(0,1)} G(x,y)\,dy \quad \text{solves}$$

$$\begin{cases} -\Delta h = 1 & \text{on } B(0,1) \\ h = 0 & \text{on } \partial B(0,1) \end{cases} \Rightarrow \quad h = \frac{-|x|^2}{n} + \frac{1}{n}$$

while

$$\ell(x) = \int_{\partial B(0,1)} K(x,y)\,dy \quad \text{solves}$$

$$\begin{matrix} -\Delta \ell = 0 & \text{on } B(0,1) \\ h = 1 & \text{on } \partial B(0,1) \end{matrix} \Rightarrow \quad \ell(x) = 1$$

Observing That

$$|h(x)| \leq \frac{1}{n}$$

We geT

$$|u(x)| \leq \max_{\partial B(0,1)} g + \frac{1}{n} \max_{B(0,1)} f.$$

n 6

From The mean value principle we Know that

$$u(0) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,r)} u(y)\, dS(y)$$

From Poisson formula we have

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)\, r} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\, dS(y)$$

Since $u \geq 0$ we have

$$\inf_{\partial B(0,r)} \frac{1}{|x-y|^n} \int_{\partial B(0,r)} u(y)\, dS(y) \leq \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\, dS(y) \leq$$

$$\leq \sup_{\partial B(0,r)} \frac{1}{|x-y|^n} \int_{\partial B(0,r)} u(y)\, dS(y)$$

Clearly we have

$$\inf_{\partial B(0,r)} \frac{1}{|x-y|^n} = \frac{1}{(r+|x|)^n}$$

$$\sup_{\partial B(0,r)} \frac{1}{|x-y|^n} = \frac{1}{(r-|x|)^n}$$

Putting together we get

$$u(x) \leq \frac{r^2 - |x|^2}{n\alpha(n)\, r} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n}\, dS(y) \leq$$

$$\leq \frac{r^{n-2}(r^2 - |x|^2)}{(r-|x|)^n} \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,r)} u(y)\, dS(y) =$$

$$= r^{n-2} \frac{r+|x|}{(r-|x|)^{n-1}} \quad u(0)$$

and similarly for the opposite inequality.

n 9

Since $u(x)=0$ when $x_n=0$ we have that $v(x) \in C^0(B(0,1))$.

Observe that from the definition we have

$$\lim_{h\to 0^+} \frac{v(x_1,\dots,x_{n-1},h)}{h} = \lim_{h\to 0^-} \frac{v(x_1\dots x_{n-1},h)}{h}$$

so that $v \in C^1(B(0,1))$. ~~Finally~~ By a similar argument we get that

$$\partial_{x_i}\partial_{x_k} v(x)$$

are continuous and also

$$\partial^2_{x_i} v(x) \qquad i \neq k$$

are continuous. Finally, since

$$\partial^2_{x_i} \cancel{u}\, u(x_1 \dots x_{n-1}, 0) = 0$$

The condition $\Delta u = 0$ implies

$$\partial^2_{x_k} u(x_1 \dots x_{n-1}, 0) = 0$$

so that

$$v \in C^2(B(0,1))$$

and

$$\Delta v = 0 \quad \text{on } C^2(B(0,1))$$