You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. In your solution you can use only statements that were proven in class.

Name: ____________________

Question:	1	2	3	Total
Points:	20	35	50	105
Score:				

1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two σ-algebras of subsets of X. Let μ_1 be a measure defined on $\mathcal{M}_1$ and μ_2 be a measure on $\mathcal{M}_2$.

   (a) (5 points) Give a definition for the measure $\mu = \mu_1 + \mu_2$. On which σ-algebra $\mathcal{M}$ is μ defined? Show that your definition defines a measure.

   Solution: We can define $\mu(A) = \mu_1(A) + \mu_2(A)$ for every $A \in \mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$. Clearly $\mathcal{M}$ is a σ-algebra. Moreover μ is a measure on $\mathcal{M}$.

   (b) (5 points) Show that if μ_1 and μ_2 are complete than $\mu = \mu_1 + \mu_2$ is complete.

   Solution: Let B be such that $B \subset C \in \mathcal{M}$ with $\mu(C) = 0$. This implies that $\mu_1(C) = 0$ so that $B \in \mathcal{M}_1$ and $\mu_1(B) = 0$. Similarly $B \in \mathcal{M}_2$ and $\mu_2(B) = 0$. Thus $B \in \mathcal{M}$ and $\mu(B) = 0$.

   (c) (10 points) Show that if $\mu_1\big|_{\mathcal{M}}$ and $\mu_2\big|_{\mathcal{M}}$ are σ-finite than $\mu = \mu_1 + \mu_2$ is also σ-finite.

   Solution: Let $A_i \in \mathcal{M}$ be such that $\mu_1(A_i) < \infty$ and $\bigcup_i A_i = X$ and $B_i \in \mathcal{M}$ be such that $\mu_2(B_i) < \infty$ and $\bigcup_i B_i = X$. Clearly $C_{i,j} = A_i \cap B_j \in \mathcal{M}$ and $\mu(C_{i,j}) \leq \mu_1(A_i) + \mu_2(B_j) < \infty$. Moreover $\bigcup_i C_{i,j} = X$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \mathcal{N})$ be a measurable space and $f : X \to Y$ a measurable function.

   (a) (5 points) Show that $\nu(B) = \mu(f^{-1}(B))$ is a measure on $(Y, \mathcal{N})$.

   Solution: If $B_i \in \mathcal{M}$ are disjoint then

   $$f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i)$$

   so that

   $$\nu\left(\bigcup_i B_i\right) = \mu\left(\bigcup_i f^{-1}(B_i)\right) = \sum_i \mu(f^{-1}(B_i)) = \sum_i \nu(B_i)$$

   where we used that $f^{-1}(B_i) \cap f^{-1}(B_j) = \emptyset$ for $i \neq j$.

   (b) (10 points) Clearly if μ is finite than ν is finite. Is it true that if μ is σ-finite than also ν is σ-finite? Is it true that if μ is complete than ν is complete?

   Solution: Assume that $\mu(X) = \infty$ and let $f(x) = y_0$ for every $x \in X$ and a given $y_0 \in Y$. Clearly ν is not σ-finite.
   Suppose $A \subset B \in \mathcal{N}$ with $\nu(B) = 0$. This implies that $\mu(f^{-1}(B)) = 0$ and $f^{-1}(A) \subset f^{-1}(B)$ so that $f^{-1}(A) \in \mathcal{M}$ and $\mu(f^{-1}(A)) = 0$. But this does not imply that $A \in \mathcal{N}$. For example let $f(x) = x$ where $X = Y = \mathbb{R}$, $\mathcal{M}$ is the Lebesgue σ-algebra while $\mathcal{N}$ is the Borel σ-algebra.

(c) (20 points) Assume now that if $X = Y = \mathbb{R}$, $\mathcal{M} = \mathcal{N} = \mathcal{L}$ and μ is the Lebesgue measure. Assume moreover that f is an increasing function with $\sup_{x\in\mathbb{R}} f = \infty$ and $\inf_{x\in\mathbb{R}} f = -\infty$. Show that ν is a Borel-Stieltjes measure. Construct the distribution function $F(x)$ of ν. Show that F is a distribution function. (**Hint** assume first that f is continuous and strictly increasing.)

Solution: Assume first that f is continuous and strictly increasing. In this case f^{-1} is a strictly increasing continuous function and

$$f^{-1}((a,b]) = (f^{-1}(a), f^{-1}(b)]$$

so that

$$\nu((a,b]) = f^{-1}(b) - f^{-1}(a)$$

and $F(x) = f^{-1}(x)$ is the distribution of ν.

Let now f be still strictly increasing but not necessarely continuous. Then f has a countable set of discountinuity y_n. Let $a_n^{\pm} = \lim_{y\to y_n^{\pm}} f(y)$ and $I_n = [a_n, b_n]$. Clearly $f^{-1}(x)$ is well defined on $\mathbb{R}\backslash\bigcup I_n$. Define $F(x) = f^{-1}(x)$ on $\mathbb{R}\backslash\bigcup I_n$ and $F(x) = y_n$ on I_n. Clearly $F(x)$ is the distribution of ν.

Finally if $f(x)$ is not strictly increasing we have that $f^{-1}(x)$ is always a closed intervall, possibly empty. In this case we can define F has above if x is such that $f^{-1}(x)$ is empty or a single point. If $f^{-1}(x) = [a,b]$ we set $f(x) = b$. Again $F(x)$ is the distribution of ν.

3. Let A and B be two Lebesgue measurable sets in $\mathbb{R}$ with finite positive measure. Define

$$F(x) = \int \chi_A(x-t)\chi_B(t)d\mu(t)$$

where μ is the Lebesgue measure and χ_A is the characteristic function of A.

(a) (20 points) Show that F is a bounded continuous function. (**Hint:** define F_ϵ by replacing A with an open set A_ϵ with $A \subset A_\epsilon$ and $\mu(A_\epsilon) < \mu(A) + \epsilon$. Show that F_ϵ is continuous. Compute $F_\epsilon(x) - F(x)$ and take the limit $\epsilon \to 0$ ...)

Solution: Let A_ϵ be a finite union of open intervals with $\mu(A_\epsilon \Delta A) \leq \epsilon$. Let

$$F_\epsilon(x) = \int \chi_{A_\epsilon}(x-t)\chi_B(t)d\mu(t)$$

Observe first that

$$\chi_{A_\epsilon}(x-t)\chi_B(t) \leq \chi_{A_\epsilon}(x-t)$$

Moreover, by Theorem 1.21, we have

$$\int \chi_{A_\epsilon}(x-t)d\mu(t) = \mu(x - A_\epsilon) = \mu(A_\epsilon) < \infty$$

Let x_n be a sequence that converge to x. We have that $\chi_{A_\epsilon}(x_n - t) \to_{n\to\infty} \chi_{A_\epsilon}(x-t)$ if $t \notin \partial A_\epsilon$. But ∂A_ϵ is a finite set so that $\mu(\partial A_\epsilon) = 0$. Thus by Dominated Convergence we have that $F_\epsilon(x_n) \to F_\epsilon(x)$ and F_ϵ is continuous.
Observe that

$$|F_\epsilon(x) - F(x)| = \int \chi_{A_\epsilon \Delta A}(x-t)\chi_B(t)d\mu(t) \leq \int \chi_{A_\epsilon \setminus A}(x-t)d\mu(t) \leq \epsilon$$

so that F_ϵ converge uniformly to F. Thus F is continuous. Finally it clearly follows that $F \leq \mu(A)$.

(b) (20 points) Compute

$$\int F(x)d\mu(x).$$

(**Hint:** this time define F_ϵ by replacing A with an open set A_ϵ with $A \subset A_\epsilon$ and $\mu(A_\epsilon) < \mu(A) + \epsilon$ and similarly for B. Show that in this case you can use Riemann integral.)

Solution: Assume first that A and B are bounded. Thus $F(x) = 0$ for $|x|$ large enough and F is integrable.

Let A_ϵ be an open set such that $A \subset A_\epsilon$ and $\mu(A_\epsilon) < \mu(A) + \epsilon$ and B_ϵ be an open set such that $B \subset B_\epsilon$ and $\mu(B_\epsilon) < \mu(B) + \epsilon$. Let

$$F_\epsilon(x) = \int \chi_{A_\epsilon}(x-t)\chi_{B_\epsilon}(t)d\mu(t).$$

Observe that $\chi_{A_\epsilon}(x-t)\chi_{B_\epsilon}(t)$ has a countable set of discontinuity so that

$$F_\epsilon(x) = \int_{-\infty}^{\infty} \chi_{A_\epsilon}(x-t)\chi_{B_\epsilon}(t)dt.$$

We know that F is continuous so that

$$\int F_\epsilon(x)d\mu(x) = \int_{-\infty}^{\infty} F_\epsilon(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{A_\epsilon}(x-t)\chi_{B_\epsilon}(t)dtdx = \mu(A_\epsilon)\mu(B_\epsilon)$$

by a simple change of variables.

In this case we have

$$F_\epsilon(x) - F(x) = \int \chi_{A_\epsilon\setminus A}(x-t)\chi_{B_\epsilon}(t)d\mu(t) + \int \chi_A(x-t)\chi_{B_\epsilon\setminus B_\epsilon}(t)d\mu(t) \leq 2\epsilon$$

and $F_\epsilon(x) > F(x)$ so that, by Monotone Convergence,

$$\int F(x)d\mu(x) = \lim_{n\to\infty} \int F_{\frac{1}{n}}(x)d\mu(x) = \mu(A)(\mu(B)$$

For generic A and B, approximate them by inersecting with $[-n, n]$.

(c) (10 points) Let $A + B = \{x + y \,|\, x \in A,\, y \in B\}$. Show that $A + B$ contains a non empty open interval. (**Hint:** Estimate χ_{A+B} using F.)

Solution: Observe that, if $F(x) > 0$ then there exists at least one t such that $\chi_A(x - t)\chi_B(t) > 0$. This implies that $x \in A + B$ and $\chi_{A+B}(x) = 1$. Thus we have that

$$\chi_{A+B}(x) \geq \frac{F(x)}{\mu(A)}.$$

From $\int F(x)d\mu(x) > 0$ we know that F is not identically 0. Thus there exists x_0 such that $F(x_0) > 0$. Since F is continuous, there exists an open interval U containing x_0 such that $F(x) > 0$ for every $x \in U$. It follow that $\chi_{A+B}(x) = 1$ for every $x \in U$, that is $U \subset A + B$.