Write your name clearly on all sheets of paper you will turn in and possibly number them. Write clearly and large enough to be easily readable. Your proofs must be **complete** and clearly written. All scalars appearing in the exercises are supposed to be real if not differently stated.
Each question is worth the number of points indicated. Fifty points will grant you an A.

1 Let $f(x)$ be a bounded measurable function from $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$\limsup |f(x)|\|x\|^\alpha = C < \infty$$

where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

a) (7 pts) For which values of α can you say that $f \in L^1(\mathbb{R}^n)$.

b) (5 pts) Is your condition on α necessary?

2 Given $f \in L^1(\mathbb{R})$ define $F(x) = \int_{-\infty}^x f(y)dy$.

a) (7 pts) Show that $F(x)$ is a continuous function. Is $F(x)$ differentiable?

b) (10 pts) Show that for every $g \in C_0^1(\mathbb{R})$ we have:

$$\int_{\mathbb{R}} F(x)g'(x)dx = -\int_{\mathbb{R}} f(x)g(x)dx.$$

Remember that $C_0^1(\mathbb{R})$ is the set of all function in $C^1(\mathbb{R})$ that are 0 outside a compact set. Moreover $C^1(\mathbb{R})$ is the set of all function that are continuous and admit a continuous derivative.

3 Given $f : [0, 1] \to \mathbb{R}$ define the *essential sup* of f as:

$$\operatorname{ess\,sup} f = \inf\{M \mid |f(x)| \le M \ a.e.\}.$$

a) (7 pts) Show that $\left|\int_0^1 f(x)dx\right| < \operatorname{ess\,sup} f$.

b) (15 pts) Show that if f is continuous then $\operatorname{ess\,sup} f = \sup f$. (**Hint**: One direction is trivial. For the other use that f is uniformly continuous because $[0, 1]$ is compact so that if $f(x) < M$ a.e. then $f(x) \le M$ everywhere.)

4 Let $(X, \mathcal{M}, \mu)$ be a measure space.

a) (10 pts) Show that if f_n is a Cauchy sequence in $L^1(X, \mu)$ than there is f measurable such that f_n converges to f in measure.

b) (10 pts) Show that if f_n is a Cauchy sequence in $L^1(X,\mu)$ and it converges almost everywhere to a function $f \in L^1(X,\mu)$ then it converges to f in $L^1(X,\mu)$. (**Hint** use Fatou lemma on $\int |f_n - f_m|$ taking the liminf on m.)

c) (10 pts) Show that if f_n is a Cauchy sequence in $L^1(X,\mu)$ than there is f in $L^1(X,\mu)$ such that f_n converges to f in $L^1(X,\mu)$.

5 (15 pts) Assume that f_n and f are such that

$$\int_{\mathbb{R}} |f_n(x)| g(x) dx < \infty \qquad \int_{\mathbb{R}} |f(x)| g(x) dx < \infty$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)| g(x) dx = 0$$

where $g \in L^+(\mathbb{R}) \cap L^1(\mathbb{R})$. Prove that for every $\epsilon > 0$ there exists E such that

$$\int_E g(x) dx < \epsilon$$

and f_n converge uniformly to f on E^c.

6 (15 pts) Let $f \in L^1(X,\mu)$. Show that for every ϵ there exists δ such that if $\mu(E) \le \delta$ then

$$\int_E f(x) d\mu(x) \le \epsilon.$$

(**Hint**: assume that there is ϵ such that for every δ you can find an E_δ with $\mu(E_\delta) \le \delta$ and $\int_{E_\delta} f(x) d\mu(x) > \epsilon$. Choose a suitable sequence of values for δ and apply the dominated convergence theorem.)