MATH 6337 EXAM 1 September 25, 2007

PLEASE READ THESE DIRECTIONS: Answer PROBLEM 1 (14 points) and choose TWO other problems to answer (10 points each). You may also answer (for up to 3 points extra credit) ONE additional problem. In this case, please specify which problem is the extra credit problem.

All statements require proof or justification. There are 34 points total, plus 1 free point for writing your name, plus up to 3 points of extra credit.

1. Note: The parts of this problem are not related.

a. Show that every σ-finite measure is semifinite.

b. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $\{x\} \in \mathcal{M}$ for every $x \in X$. Show that if μ is a finite measure, then

$$E \;=\; \{x \in X : \mu\{x\} > 0\}$$

is countable.

Hint: $E = \cup E_n$ where $E_n \;=\; \{x \in X : \mu\{x\} > \frac{1}{n}\}$. How many points can there be in E_n?

2. Let

$$\mathcal{E} \;=\; \{[a, \infty) : a \in \mathbb{R}\},$$

and let $\mathcal{M}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E}$. Give a complete proof that $\mathcal{M}(\mathcal{E}) = \mathcal{B}$, the Borel σ-algebra on $\mathbb{R}$ (that is, you can't just say that we said in class that this is true, you have to prove it).

3. Give the definition of Lebesgue exterior measure on $\mathbb{R}^d$, and prove directly from this definition that Lebesgue exterior measure is monotonic and countably subadditive.

4. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a *Lipschitz function*, which means that there exists a constant $K > 0$ such that

$$\forall\, x, y \in \mathbb{R}, \quad |f(x) - f(y)| \;\leq\; K\,|x - y|.$$

Prove that if $Z \subseteq \mathbb{R}$ has Lebesgue measure $|Z| = 0$, then $f(Z) = \{f(x) : x \in Z\}$ also has Lebesgue measure zero.

5. Let $F\colon \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function, and let $\mu = \mu_F$ be the corresponding Lebesgue–Stieltjes measure. Show that

$$\mu\{x\} \;=\; 0 \text{ for all } x \in \mathbb{R} \qquad \Longleftrightarrow \qquad F \text{ is continuous.}$$