polynomials.

8. Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all z in G. Show that either $f \equiv 0$ or $g \equiv 0$.

9. Let $U: \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all z in $\mathbb{C}$; prove that U is constant.

10. Show that if f and g are analytic functions on a region G such that $\bar{f}g$ is

integer k there is a point $a \notin \{\gamma\}$ with $n(\gamma, a) = k$.

3. Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that $\int_\gamma \frac{p'(z)}{p(z)}\, dz = 2\pi i n$.

Exercises

1. Suppose $f: G \to \mathbb{C}$ is analytic and define $\varphi: G \times G \to \mathbb{C}$ by $\varphi(z,w) = [f(z) - f(w)](z-w)^{-1}$ if $z \neq w$ and $\varphi(z,z) = f'(z)$. Prove that φ is continuous and for each fixed w, $z \to \varphi(z,w)$ is analytic.
2. Give the details of the proof of Theorem 5.6.
3. Let $B_{\pm} = \bar{B}(\pm 1; \frac{1}{2})$, $G = B(0;3) - (B_+ \cup B_-)$. Let $\gamma_1, \gamma_2, \gamma_3$ be curves whose traces are $|z-1| = 1$, $|z+1| = 1$, and $|z| = 2$, respectively. Give γ_1, γ_2, and γ_3 orientations such that $n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0$ for all w in $\mathbb{C} - G$.
4. Show that the Integral Formula follows from Cauchy's Theorem.
5. Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$ $\int_\gamma (z-a)^{-n}\, dz = 0$.
6. Let f be analytic on $D = B(0;1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.
7. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Find $\int_\gamma \left(\frac{z}{z-1}\right)^n dz$ for all positive integers n.
8. Let G be a region and suppose $f_n: G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f: G \to \mathbb{C}$ Show that f is analytic.
9. Show that if $f: \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$ then f is an entire function.
10. Use Cauchy's Integral Formula to prove the Cayley–Hamilton Theorem: If A is an $n \times n$ matrix over $\mathbb{C}$ and $f(z) = \det(z - A)$ is the characteristic polynomial of A then $f(A) = 0$. (This exercise was taken from a paper by C. A. McCarthy, *Amer. Math. Monthly*, **82** (1975), 390–391).

1. Let G be a region and let $\sigma_1, \sigma_2\colon [0,1] \to G$ be the constant curves $\sigma_1(t) \equiv a$, $\sigma_2(t) \equiv b$. Show that if γ is a closed rectifiable curve in G and $\gamma \sim \sigma_1$ then $\gamma \sim \sigma_2$. (Hint: connect a and b by a curve.)

2. Show that if we remove the requirement "$\Gamma(0, t) = \Gamma(1, t)$ for all t" from Definition 6.1 then the curve $\gamma_0(t) = e^{2\pi i t}$, $0 \le t \le 1$, is homotopic to the constant curve $\gamma_1(t) \equiv 1$ in the region $G = \mathbb{C} - \{0\}$.

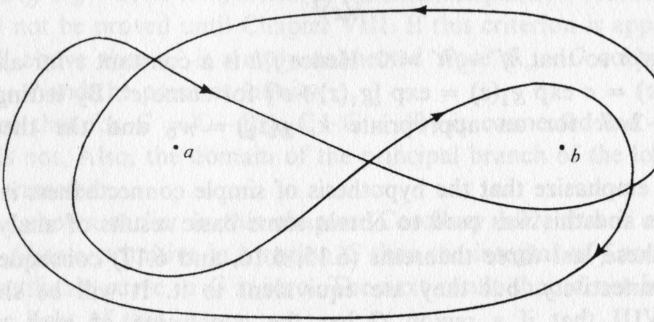
3. Let $\mathscr{C}$ = all rectifiable curves in G joining a to b and show that Definition 6.11 gives an equivalence relation on $\mathscr{C}$.

4. Let $G = \mathbb{C} - \{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z\colon |z| = 1\}$.

6. Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \le \theta \le 4\pi$. Evaluate $\displaystyle\int_\gamma \frac{dz}{z^2 + \pi^2}$.

7. Let $f(z) = [(z - \frac{1}{2} - i) \cdot (z - 1 - \frac{3}{2}i) \cdot (z - 1 - \frac{i}{2}) \cdot (z - \frac{3}{2} - i)]^{-1}$ and let γ be the polygon $[0, 2, 2+2i, 2i, 0]$. Find $\int_\gamma f$.

8. Let $G = \mathbb{C} - \{a, b\}$, $a \ne b$, and let γ be the curve in the figure below.



(a) Show that $n(\gamma; a) = n(\gamma; b) = 0$.

(b) Convince yourself that γ is not homotopic to zero. (Notice that the word is "convince" and not "prove". Can you prove it?) Notice that this example shows that it is possible to have a closed curve γ in a region such that $n(\gamma; z) = 0$ for all z not in G without γ being homotopic to zero. That is, the converse to Corollary 6.10 is false.

9. Let G be a region and let γ_0 and γ_1 be two closed smooth curves in G. Suppose $\gamma_0 \sim \gamma_1$ and Γ satisfies (6.2). Also suppose that $\gamma_t(s) = \Gamma(s, t)$ is smooth for each t. If $w \in \mathbb{C} - G$ define $h(t) = n(\gamma_t; w)$ and show that $h: [0, 1] \to \mathbb{Z}$ is continuous.

10. Find all possible values of $\displaystyle\int_\gamma \frac{dz}{1 + z^2}$ where γ is any closed rectifiable curve in $\mathbb{C}$ not passing through $\pm i$.

3. Let f be analytic in $B(a; R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = \ldots = f(a) = 0$ and $f^{(m)}(a) \neq 0$.

4. Suppose that $f: G \to \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any z in G.