CONTRACTION MAPPING THEOREM WITH PARAMETERS

S. SCHECTER

Let U be an open subset of a Banach space X, V an open subset of a Banach space Y, $T : U \times V \to V$ a continuous map. T is a *uniform contraction* if there is number λ, $0 < \lambda < 1$, such that for all $x \in U$ and for all $y, y' \in V$, $|T(x,y) - T(x,y')| \leq \lambda |y - y'|$.

Theorem 1. *Let $T : U \times V \to V$ be a uniform contraction. Let $g(x)$ be the unique fixed point of the mapping $T(x, \cdot)$ from V to V. Then:*

(1) *g is continuous.*
(2) *If T is C^1 then g is C^1 and $Dg(x) = (I - D_2T(x, g(x)))^{-1} D_1 T(x, g(x))$.*

In a homework problem we show that the uniform contraction assumption implies that $\|D_2T(x,y)\| \leq \lambda$ for all (x,y). Since $\lambda < 1$, $I - D_2T(x,y)$ is invertible.

Note that the equation $g(x) = T(x, g(x))$ implies that *if* T is differentiable, then $Dg(x)$ is given by the formula. Also, once we know that $Dg(x)$ is given by the formula, the formula implies that g is C^1.

Proof. The steps are:

(1) g is continuous.
(2) If T is C^1, then g is locally Lipschitz.
(3) If T is C^1, then g is differentiable, and $Dg(x)$ is given by the formula.

By the above remark, step 3 implies that g is C^1.

1. g is continuous: We have

$$\begin{aligned} |g(x') - g(x)| &= |T(x', g(x')) - T(x, g(x))| \\ &\leq |T(x', g(x')) - T(x', g(x))| + |T(x', g(x)) - T(x, g(x))| \\ &\leq \lambda |g(x') - g(x)| + |T(x', g(x)) - T(x, g(x))|. \end{aligned}$$

Therefore

$$|g(x') - g(x)| \leq (1-\lambda)^{-1} |T(x', g(x)) - T(x, g(x))|. \tag{1}$$

Now fix $x \in U$. Because T is continuous,

given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } |x' - x| < \delta \text{ then } |T(x', g(x)) - T(x, g(x))| < (1-\lambda)\epsilon. \tag{2}$$

By (2) and (1), if $|x' - x| < \delta$ then $|g(x') - g(x| < \epsilon$. Therefore g is continuous at x. Since x was arbitrary, g is continuous.

2. If T is C^1, then g is locally Lipschitz: Assume T is C^1. Let $x_0 \in U$. Choose $\epsilon > 0$ and $C > 0$ such that if $|x - x_0| < \epsilon$ and $|y - g(x_0)| < \epsilon$ then $(x, y) \in U \times V$ and $\|D_1T(x,y)\| \leq C$. Then choose δ, $0 < \delta < \epsilon$, such that if $|x - x_0| < \delta$ then $|g(x) - g(x_0)| < \epsilon$. (This is possible because we showed g is continuous.) Let $|x - x_0| < \delta$ and $|x' - x_0| < \delta$. Using (1),

$$|g(x') - g(x)| \leq (1-\lambda)^{-1} |T(x', g(x)) - T(x, g(x))| \leq (1-\lambda)^{-1} C |x' - x|.$$

Date: January 19, 2013.

Thus on the δ-neighborhood of x_0, g is Lipschitz with Lipschitz constant $(1-\lambda)^{-1}C$. Since x_0 is arbitrary, g is locally Lipschitz.

3. If T is C^1 then g is C^1 and $Dg(x) = (I - D_2T(x, g(x)))^{-1}D_1T(x, g(x))$: Let $A(x) = (I - D_2T(x, g(x)))^{-1}D_1T(x, g(x))$, i.e., $A(x)$ is the unique solution of the equation

$$A = D_1T(x, g(x)) + D_2T(x, g(x))A.$$

We must show that $Dg(x) = A(x)$.

$$\begin{aligned}
g(x+h)-g(x)-A(x)h &= T(x+h, g(x+h))-T(x, g(x))-(D_1T(x, g(x))+D_2T(x, g(x))A(x))h \\
&= T(x+h, g(x+h)) - T(x+h, g(x)) - D_2T(x, g(x))A(x))h \\
&\quad + T(x+h, g(x)) - T(x, g(x)) - D_1T(x, g(x))h \\
&= \left(\int_0^1 D_2T\big(x+h, g(x)+s(g(x+h)-g(x))\big)\,ds\right)(g(x+h))-g(x))-D_2T(x, g(x))A(x))h \\
&\quad + \left(\int_0^1 D_1T(x+sh, g(x))\,ds\right)h - D_1T(x, g(x))h \\
&= D_2T(x, g(x))\big(g(x+h)-g(x)-A(x)h\big) \\
&\quad + \left(\int_0^1 D_2T\big(x+h, g(x)+s(g(x+h)-g(x))\big) - D_2T(x, g(x))\,ds\right)(g(x+h))-g(x)) \\
&\quad + \left(\int_0^1 D_1T(x+sh, g(x)) - D_1T(x, g(x))\,ds\right)h
\end{aligned}$$

Now we estimate $|g(x+h)-g(x)-A(x)h|$ the way we estimated $|g(x')-g(x)|$ in (1), and we estimate the two integrals. We get

$$\begin{aligned}
&|g(x+h)-g(x)-A(x)h| \\
&\le (1-\lambda)^{-1}\Bigg(\left(\sup_{0\le s\le 1}\|D_2T\big(x+h, g(x)+s(g(x+h)-g(x))\big) - D_2T(x, g(x))\|\right)(g(x+h))-g(x)) \\
&\qquad + \left(\sup_{0\le s\le 1} D_1T(x+sh, g(x)) - D_1T(x, g(x))\,ds\|\right)|h|\Bigg).
\end{aligned}$$

Since g is locally Lipschitz by step 2, $|g(x+h))-g(x)| \le C|h|$. For $|h|$ sufficiently small, the two sups can be made as small as we want, say $\le \epsilon$. Then for h small,

$$|g(x+h)-g(x)-A(x)h| \le (1-\lambda)^{-1}(\epsilon C + \epsilon)|h|.$$

Therefore

$$\lim_{h\to 0}\frac{g(x+h)-g(x)-A(x)h}{|h|} = 0,$$

so $Dg(x) = A(x)$.

$\square$