1.3. Uniqueness, continuous dependence and differentiability.

In this section, we prove uniqueness of the solution of the initial value problem (1.2) under the hypotheses that f is locally Lipschitzian and give also some differentiability results when the function f satisfies additional regularity properties.

A function $f \in C^0(D, \mathbb{R}^d)$ is said to be *locally Lipschitzian* with respect to x if, for any closed bounded set $U \subset D$, there is a constant $k = k_U$ (called the *Lipschitz constant on* U) such that $|f(t, x) - f(t, y)| \le k|x - y|$ for (t, x), $(t, y) \in U$. We let $C^r_x(D, \mathbb{R}^d)$ be the set of functions which have continuous derivatives with respect to x up through order r and let $C^{r,1}_x(D, \mathbb{R}^d)$ be those functions in $C^r_x(D, \mathbb{R}^d)$ which have r^{th} derivatives locally Lipschitzian.

Theorem 3.1. *If $f \in C^r_x(D, \mathbb{R}^d), r \ge 1$, then, for any $(\tau, \xi) \in D$, there is a unique solution $x(t, \tau, \xi)$ of the initial value problem (1.2) and $x(t, \tau, \xi)$ is continuous in (t, τ, ξ) together with first derivatives with respect to t, τ and all derivatives with respect to ξ up through order r. If $f \in C^{r,1}_x(D, \mathbb{R}^d)$, $r \ge 0$, then, for any $(\tau, \xi) \in D$, there is a unique solution $x(t, \tau, \xi)$ of the initial value problem (1.2) and $x(t, \tau, \xi)$ is continuous in (t, τ, ξ) together with all derivatives with respect to ξ up through order r with the r^{th} derivatives locally Lipschitzian.*

Proof. For any given closed bounded subset U of D, we choose positive constants $\bar{\alpha}$, $\bar{\beta}$ so that the rectangle $R = R(\tau, \xi)$ in the proof of Theorem 1.1 belongs to D for all $(\tau, \xi) \in U$ and so that $V = \cup\{ R(\tau, \xi) : (\tau, \xi) \in U \}$ has its closure in D.If $M = \sup\{ |f(t, x)| : (t, x) \in V \}$, and k is the Lipschitz constant of f with respect to x or V, then we choose positive constants $\alpha \le \bar{\alpha}$, $\beta \le \bar{\beta}$ so that $M\alpha \le \beta$, $k\alpha < 1$. If $\Gamma = \{ \varphi \in C^0([-\alpha, \alpha], \mathbb{R}^d) : \varphi(0) = 0, |\varphi(t)| \le \beta \text{ for } t \in [-\alpha, \alpha] \}$, then we define the map $T : \Gamma \to C^0([-\alpha, \alpha], \mathbb{R}^d)$ by the relation

$$T\varphi(t) = \int_{\tau}^{t+\tau} f(s, \varphi(s - \tau) + \xi)\, ds\,. \tag{3.1}$$

The fixed points of T in Γ coincide with the solutions $x(t, \tau, \xi) = \varphi(t - \tau) + \xi$ of the initial value problem (1.2) on $[\tau - \alpha, \tau + \alpha]$.

We now show that T is a uniform contraction on Γ (see Section A.2). We observe first that Γ is a closed subset of the Banach space $C^0([-\alpha, \alpha], \mathbb{R}^d)$. Since $T\varphi(0) = 0$ and an easy computation shows that $|T\varphi(t)| \le M\alpha \le \beta$ for $t \in [-\alpha, \alpha]$, we have $T\Gamma \subset \Gamma$. Also, $|T\varphi(t) - T\psi(t)| \le k\alpha \sup_{s\in[-\alpha,\alpha]}$ for $t \in [-\alpha, \alpha]$. Since $k\alpha < 1$, this shows that T is a uniform contraction on Γ and the conclusion in the theorem follows from Theorem A.2.1.

Corollary 3.1. *Suppose that $f \in C^{r,1}_x(D, \mathbb{R}^d), r \ge 0$. If $x(t, \tau_0, \xi_0)$ is a solution of (1.2) defined on the maximal interval (α_0, β_0), then, for any closed interval $[a, b] \subset (\alpha_0, \beta_0)$, there is a $\delta = \delta([a, b], \tau_0, \xi_0)$ such that, for any (τ, ξ) with $|\tau - \tau_0| < d, |\xi - \xi_0| < \delta$, the solution $x(t\tau, \xi)$ of (1.2) exists on $[a, b]$ and $x(t, \tau, \xi) \to x(t\tau_0, \xi_0)$ uniformly on $[a, b]$ as $(\tau, \xi) \to (\tau_0, \xi_0)$.*

Proof. This is a consequence of Theorem 3.1 since $x(t, \tau, \xi)$ is uniformly continuous on compact sets.

Since the contraction principle was used in the proof of Theorem 3.1, we can obtain the solution by successive approximations

$$\varphi^{(n+1)} = T\varphi^{(n)}, \quad n = 0, 1 \ldots$$

where T is defined in (3.1) and $\varphi^{(0)}$ is any function belonging to Γ. The simplest choice for $\varphi^{(0)}$ is the zero function. If we return to the original variable x, this class of successive approximations is given by

$$\begin{aligned} &x^{(0)} = \xi, \quad x^{(n+1)} = \tilde{T}x^{(n)}, \quad n = 0, 1 \ldots \\ &\tilde{T}x(t) = \xi + \int_\tau^t f(s, x(s))\, ds\,. \end{aligned} \tag{3.2}$$

Exercise 3.1. (*Successive approximations converge*) Prove directly that the successive approximations (3.2) converge for $M\alpha \leq \beta$, where M, α, β are the constants chosen in the proof of Theorem 3.1.

Exercise 3.2. (*Approximations and Taylor series*) Apply the successive approximations to the scalar initial value problem $\dot{x} = -x$, $x(0) = 1$, to obtain

$$x^{(n)}(t) = 1 - t + \ldots + (-1)^n \frac{t^n}{n!}\,,$$

which is the truncated Taylor expansion for e^{-t}.

We often need the formulas for the derivatives of $x(t, \tau, \xi)$ with respect to τ, ξ. It is easy to verify that each column of the $d \times d$ matrix

$$\frac{\partial x(t, \tau, \xi)}{\partial \xi}, \quad \frac{\partial x(\tau, \tau, \xi)}{\partial \xi} = I, \text{ the identity}$$

satisfies the *linear variational equation*

$$\dot{y} = \frac{\partial f(t, x(t, \tau, \xi))}{\partial x}\, y\,. \tag{3.3}$$

We also can show that

$$\frac{\partial x(t, \tau, \xi)}{\partial \tau} = -\frac{\partial x(t, \tau, \xi)}{\partial \xi} f(\tau, \xi)\,. \tag{3.4}$$

In fact, from the uniqueness of the solution, for any real h sufficiently small, we have $x(t, \tau, \xi) = x(t, \tau + h, x(\tau + h, \tau, \xi))$ since they both satisfy the same differential equation and are equal at $t = \tau + h$. Therefore,

$$\begin{aligned} &x(t, \tau + h, \xi) - x(t, \tau, \xi) \\ &\qquad = x(t, \tau + h, \xi) - x(t, \tau + h, x(\tau + h, \tau, \xi)) \,. \end{aligned}$$

Dividing by h and taking the limit as $h \to 0$ yields (3.4).

The linear variational arises also in the following important way. If $\psi(t)$ is a solution of (1.1) and we are interested in the behavior of the solutions of (1.1) in a neighborhood of this given solution, then the transformation $x = y + \psi(t)$ yields a new differential equation for which $y = 0$ is a solution. The linear terms in the expansion of the vector field about $y = 0$ gives the linear variational equation.

Theorem 3.2. *(Analyticity in initial data) If $f : \mathbb{C}^d \to \mathbb{C}^d$ is an analytic function, then the solution $x(t, 0, \xi)$ is analytic in ξ.*

Exercise 3.3. Prove Theorem 3.2.

Theorem 3.3. *(Regularity in time) If $f \in C^r(D, \mathbb{R}^d)$, $r \geq 1$, then the solution $x(t, \tau, \xi)$ of the initial value problem (1.2) is C^r in all of its arguments.*

Exercise 3.4. Prove Theorem 3.3. *Hint*: Let $\dot{t} = 1$ and consider the initial value problem for $z = (x, t)$.

Theorem 3.4. *(Dependence on parameters) For the equation*

$$\dot{x} = f(x, \lambda)$$

where $f \in C^r(\mathbb{R}^d \times \mathbb{R}^k)$, $r \geq 1$, the solution $x(t, \xi, \lambda)$, $x(0, \xi, \lambda) = \xi$, is a C^r-function of its arguments. Furthermore, for any $\mu \in \mathbb{R}^k$, the function $\frac{\partial}{\partial \lambda} x(t, \xi, \lambda)\mu$ is a solution of the initial value problem

$$\begin{aligned} \dot{z} &= \frac{\partial}{\partial x} f(x(t, \xi, \lambda), \lambda) z + \frac{\partial}{\partial \lambda} f(x(t, \xi, \lambda), \lambda)\mu \\ z(0) &= 0 \,. \end{aligned}$$

Exercise 3.5. Prove Theorem 3.4. *Hint*: Put $\dot{\lambda} = 0$ and consider the differential equation for $z = (x, \lambda)$.

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