Exercise 1

Consider the differential equation

$$\dot{x} = f(x,t) \tag{1}$$

with initial condition $x(t_0) = x_0$. Assume that $f \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$. Given $h > 0$ we call $x^h(t)$ (Euler approximation) the function defined by

$$\begin{cases} x^h(nh+t) = x^h(nh) + f(x^h(nh), nh)t & \text{for } n \geq 0 \text{ and } 0 \leq t \leq h \\ x^h(nh+t) = x^h(nh) + f(x^h(nh), nh)t & \text{for } n \leq 0 \text{ and } -h \leq t \leq 0 \end{cases} \tag{2}$$

Prove existence and uniqueness of the solution of eq.(1) using the Euler approximations. Show how it happens that, if the function f is not Lipschitz, the solution may fail to be unique.

Exercise 2

Let $x(t)$ be a solution of

$$\dot{x} = f(x) \tag{3}$$

with $x(0) = x_0$ and $x(1) = x_1$. Call γ the trajectory $\{x(t),\ t \in [0,1]\}$. Assume that $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. Let $h_0(x)$ and $h_1(x)$ to smooth function from $\mathbb{R}^n$ in $\mathbb{R}$ such that

$$h_0(x_0) = 0 \qquad\qquad h_1(x_1) = 0 \tag{4}$$

Under which condition the equations:

$$h_0(x) = 0 \qquad\qquad h_1(x) = 0 \tag{5}$$

define two $(n-1)$-cells S_0 and S_1 transverse to γ?

Under these condition, show that there is a differentiable function F from a small neighbor of x_0 on S_0 to a small neighbor of x_1 in S_1 such that $F(x)$ is on the trajectory of eq.(3) starting from x. Compute

$$\frac{\partial F}{\partial x}(x) \tag{6}$$

Exercise 3

Consider the differential equation

$$\begin{cases} \dot{x} = -y + \epsilon f_x(x,y) \\ \dot{y} = x + \epsilon f_y(x,y) \end{cases} \tag{7}$$

where $f = (f_x, f_y)$ is a smooth function from $\mathbb{R}^2$ in $\mathbb{R}^2$ and ϵ is a small parameter. Call $\phi(\xi, t)$ the solution of eq.(7) starting at ξ at time 0. Let $\xi = (x, 0)$, $x > 0$, be a point on the positive x axis. Show that if ϵ is small enough, there is a time $t_\epsilon(x)$ close to 2π such that $\phi((x,0), t(x))$ is again on the positive x axis.

Call $F_\epsilon(x)$ the map define by $F_\epsilon(x) = \phi_x((x,0), t(x))$ where $\phi(\xi, t) = (\phi_x(\xi, t), \phi_y(\xi, t))$. Show that, for ϵ small enough, F_ϵ is a smooth map from a neighbor of x in $\mathbb{R}$ to a neighbor of $F_\epsilon(x)$ in $\mathbb{R}$. Compute

$$\partial_\epsilon F_\epsilon(x) = \frac{\partial F_\epsilon}{\partial \epsilon}(x) \tag{8}$$

by treating ϵ as a parameter. Show that if there are x_1 and x_2, $x_1 < x_2$, such that $\partial_\epsilon F_\epsilon(x_1) > 0 > \partial_\epsilon F_\epsilon(x_2)$ then there is a periodic orbit starting from some point $(\bar{x}, 0)$ with $x_1 \le \bar{x} \le x_2$.