2.8.2. Affine Maps.

In this section, we discuss affine maps in the same spirit as we have discussed nonhomogeneous linear systems in Section 2.6.

Let L be a $d \times d$ complex matrix and let $\{ g_k \}_{k=0}^{\infty}$ be a given sequence of complex d-vectors. For each integer $n \geq 0$, we consider the map

$$L + g_n : \mathbb{C}^d \to \mathbb{C}^d \quad x \mapsto Lx + g_n \,.$$

The objective is to understand the behavior of the sequence $\{ x_n \}_{n=0}^{\infty}$ defined by

$$x_n = Lx_{n-1} + g_{n-1}, \quad n \geq 1, \quad x_0 \text{ arbitrary}\,. \tag{8.5}$$

Example 8.3. Consider the nonhomogeneous differential equation $\dot{x} = Ax + g(t)$, where A is a $d \times d$ constant matrix and g is a continuous d-vector. The variation of constants formula for this equation is

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-s)}g(s)\,ds = e^{A(t-\tau)}x(\tau) + \int_{0}^{t-\tau} e^{A(t-\tau-s)}g(s+\tau)\,ds.$$

If we define $x(n) = x_n$, $n \geq 0$, and take $t = n$, $\tau = n-1$, then, for $n \geq 1$, we have

$$x_n = e^{A}x_{n-1} + \int_{0}^{1} e^{A(1-s)}g(s+n-1)\,ds \equiv Lx_{n-1} + g_{n-1}\,,$$

where $L = e^A$, $g_{n-1} = \int_0^1 e^{A(1-s)}g(s+n-1)\,ds$. In this way, we obtain a special case of a map (8.5). This map is called the *time one map* of the differential equation and the point x_0 is the initial condition.

Exercise 8.3. Suppose that g is a continuous 1-periodic d-vector function, A is a constant $d \times d$ matrix and consider the equation $\dot{x} = Ax + g(t)$. Show that the time one map is given by $L + g_0$, where $L = e^A$ and $g_0 = \int_0^1 e^{A(1-s)}g(s)\,ds$. Recall that fixed points of $L + g_0$ correspond to 1-periodic solutions of the differential equation. If the orthogonality condition in the Fredholm Alternative is satisfied, then there must be a fixed point of $L + g_0$. Interpret this condition in terms of the left eigenvectors of L corresponding to the eigenvalue 1.

The analogue of the variation of constants formula for the sequence in (8.5) is to express the value of x_n in terms of the initial condition x_0, the operator L and the sequence $\{ g_n \}$. It is easy to verify that this is given by

$$x_n = L^n x_0 + \Sigma_{j=0}^{n-1} L^{n-j-1} g_j\,. \tag{8.6}$$

2.8.3. Stability and Perturbations.

In this section, we give some simple results concerned with the effects that perturbations of a linear map on $\mathbb{R}^d$ will have upon the stability of the origin. They are obtained by means of the variation of constants formula and simple inequalities.

Theorem 8.2. *Suppose that L is a real $d \times d$ matrix, M_n, $n \geq 0$, are real $d \times d$ matrices satisfying*

$$m_0 \equiv \sup\{\, |M_n|,\, n \geq 0 \,\} < \infty \tag{8.7}$$

and define the sequence

$$x_n = (L + M_{n-1})x_{n-1}, \;\; n \geq 1, \;\; x_0 \text{ arbitrary} . \tag{8.8}$$

If the origin is a global attractor for the map L, then there are constants $k > 0$, $0 \leq c < 1$, $\delta > 0$ such that, if $m_0 < \delta$, then

$$|x_n| \leq kc^n |x_0|, \;\; n \geq 1 . \tag{8.9}$$

Proof. If we consider the term $M_{n-1}x_{n-1}$ as a nonhomogeneous term, then the formula (8.6) yields

$$x_n = L^n x_0 + \Sigma_{j=0}^{n-1} L^{n-j-1} M_j x_j, \;\; n \geq 1 .$$

Since the origin is a global attractor for L, it follows from Lemmas 8.3 and 8.2 that there are constants $k_1 \geq 1$, $0 \leq c_1 < 1$, such that $|L^n| \leq k_1 c_1^n$ for $n \geq 0$. Thus, we can obtain the following estimate for $n \geq 1$:

$$\begin{aligned} |x_n| &\leq k_1 c_1^n |x_0| + \Sigma_{j=0}^{n-1} k_1 c_1^{n-1-j} |M_j|\, |x_j| \\ &\leq k_1 c_1^n |x_0| + \Sigma_{j=0}^{n-1} m_0 k_1 c_1^{n-1-j} |x_j| . \end{aligned}$$

If $c_1 = 0$, this inequality is $|x_n| \leq m_0 k_1 |x_{n-1}|$, $n \geq 1$, and so $|x_n| \leq (m_0 k_1)^n |x_0|$. If $m_0 k_1 < 1$, we have the conclusion in the theorem by taking any fixed $\delta < k_1^{-1}$, $k = 1$, $c = \delta k_1$. Let us now assume that $c_1 > 0$. If we let $y_j = c_1^{-j}|x_j|$, $j \geq 0$, then we have

$$y_n \leq k_1 y_0 + k_1 m_0 c_1^{-1} \Sigma_{j=0}^{n-1} y_j .$$

If we define a sequence $\{\, z_n,\, n \geq 0, \,\}$ be replacing the inequality by equality and let $z_0 = y_0$, $z_1 = k_1 z_0 + k_1 m_0 c_1^{-1} z_0$, then $y_n \leq z_n$ for all $n \geq 0$. The sequence $\{\, z_n,\, n \geq 2 \,\}$ satisfies the relation

$$\begin{aligned} z_n &= k_1 z_0 + k_1 m_0 c_1^{-1} \Sigma_{j=0}^{n-1} z_j \\ &= z_{n-1} + k_1 m_0 c_1^{-1} z_{n-1} = (1 + k_1 m_0 c_1^{-1}) z_{n-1} . \end{aligned}$$

This implies that

$$z_n \leq (1 + k_1 m_0 c_1^{-1})^{n-1} z_1 \leq k_1 (1 + m_0 c_0^{-1})(1 + k_1 m_0 c_1^{-1})^{n-1} z_0$$

for $n \geq 0$. Since $y_n \leq z_n$ for all $n \geq 0$ and $y_j = c_1^{-j} x_j$ for all $j \geq 0$ and $k_1 \geq 1$, we deduce that

$$x_n \leq k_1 c_1^n (1 + k_1 m_0 c_1^{-1})^n x_0$$

for $n \geq 0$. Since $0 < c_1 < 1$, we can choose $\delta > 0$ so that $c = c_1(1 + k_1 \delta c_1^{-1}) < 1$. If $m_0 < \delta$ and $k = k_1$, then we obtain the conclusion stated in the theorem.

Exercise 8.4. For a sequence of real numbers $\alpha_n \geq 0$, $n \geq 1$, show that the infinite product $\prod_{n=1}^{\infty}(1 + \alpha_n)$ converges if the infinite sum $\Sigma_{n=1}^{\infty} \alpha_n$ converges. *Hint.* The mean value theorem shows that there exists $\theta_n \in [0, 1]$ such that

$$\log(1 + \alpha_n) = \log(1 + \alpha_n) - \log 1 = \frac{\alpha_n}{1 + \theta \alpha_n} \leq \alpha_n .$$

Exercise 8.5. Consider the mapping

$$z_n = (I + M_{n-1}) z_{n-1}, \ \ n \geq 1, \ \ z_0 \text{ arbitrary.}$$

Use the previous exercise to find sufficient conditions on the matrices M_{n-1}, $n \geq 1$, which will ensure that there is a constant k such that $|z_n| \leq k|z_0|$ for all $n \geq 0$ and all z_0.

Theorem 8.3. *(Principle of Linearization) Suppose that $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, and x_0 is a fixed point of f. If $L = \partial f(x_0)/\partial x$ and $\rho \in \sigma(L)$ implies $|\rho| < 1$, then the fixed point x_0 is a local attractor of f.*

Proof. If we replace $f(x)$ by $f(x + x_0) - x_0$, then we can assume that the fixed point is 0. Let $g(x) = f(x) - Lx$. In this case, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|g(x)| < \epsilon |x|$ if $|x| < \delta$. If $\xi \in \mathbb{R}^d$ is given and we let $x_0 = \xi$, $x_n = f^n(\xi)$, $n \geq 1$, then

$$x_n = L x_{n-1} + g(x_{n-1}) = L^n \xi + \Sigma_{j=0}^{n-1} L^{n-1-j} g(x_j) .$$

There are constants $k \geq 1$, $0 \leq c < 1$, such that (8.3) is valid. If we suppose that $|x_j| < \delta$ for $0 \leq j \leq n-1$, then we can use (8.3) to obtain the estimate

$$|x_n| \leq k c^n |\xi| + k \epsilon \Sigma_{j=0}^{n-1} c^{n-1-j} |x_j| .$$

If we let $y_j = c^{-j} |x_j|$, $j \geq 0$, then we have

$$y_n \leq k y_0 + k \epsilon \Sigma_{j=0}^{n-1} y_j .$$

Now we proceed exactly as in the proof of Theorem 8.2 letting ϵ play the role of m_0. In this way, there are constants $\epsilon_0 > 0, \delta_0 > 0$, $0 \leq c_1 < 1, k_1 > 0$, such that $|x_n| \leq k_1 c_1^n |\xi|$ as long as $|x_j| \leq \delta_0$, $j = 0, 1 \ldots, n-1$. If we choose $\delta_1 > 0$ so that $k_1 \delta_1 < \delta_0$ and take $|\xi| < \delta_1$, then we have the conclusion stated in the theorem.

2.8.4. Quadratic Liapunov functions.

For a linear autonomous ordinary differential equation for which the zero solution was an attractor (asymptotically stable), we showed that there was a positive definite quadratic form for which the derivative along the solutions was a negative definite quadratic form. In this way, we had a very geometric understanding of the meaning of the attractor in terms of the vector field crossing the level sets of the quadratic form. We show there is an analogous interpretation for the situation of a linear map with all eigenvalues of modulus < 1.

If A is a given $d \times d$ constant matrix, let $x' = Ax$ for $x \in \mathbb{R}^d$. If $V : \mathbb{R}^d \to \mathbb{R}$ is a continous function, then the analogue of the above derivative along the solutions is $V(x') - V(x)$. If $V(x) = x^* B x$ is a quadratic form, then

$$V(x') - V(x) = x^*(A^* B A - B)x.$$

If B is positive definite and this quantity is negative definite, then $A^n x \to 0$ as $n \to \infty$ (prove this); that is, the eigenvalues of A have modulus < 1. Is it possible to show that there is such a function V if the eigenvalues of A have modulus < 1? We state the following result without proof (see, for example, LaSalle, Appl. Math. Sci. **62**, p.36).

Lemma 8.3. *Suppose that A, C are given $d \times d$ matrices. The equation*

$$A^* B A - B = -C \tag{8.9}$$

has a solution if and only if A has the property that, if $\rho \neq 0$ is an eigenvalue of A, then ρ^{-1} is not an eigenvalue of A.

Now, suppose that the eigenvalues of A have modulus < 1. Then the conditions of Lemma 8.3 are satisfied. If we choose $C = I$, then there is a unique solution B of (8.9). For any $\epsilon > 0$, by a change of coordinates, we can suppose that each Jordan block for A has an ϵ in the upper diagonal. Therefore, if we choose ϵ sufficiently small, then it is clear that B is positive definite. If we let $V(x) = x^* B x$, then V serves as a Liapunov function and $V(x') - V(x) = -x^* x$, as we wanted.

Exercise 8.6. Give proofs of Theorem 8.2 and Theorem 8.3 using a quadratic Liapunov function.

Exercise 8.7. Prove the following result. *Suppose that $f \in C^r(\mathbb{R}^d, \mathbb{R}^d)$, $r \geq 1$, and x_0 is a fixed point of f. If $L = \partial f(x_0)/\partial x$ and $\rho \in \sigma(L)$ implies $|\rho| \neq 1$, and there is at least one $\rho \in \sigma(L)$ with $|\rho| > 1$, then the fixed point x_0 is unstable.*

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