2.7. The Fredholm Alternative.

When both A and g in (5.1) are p-periodic, we refer to (5.1) as a *p-periodic system.* In this section, for a p-periodic system (5.1), we use the adjoint equation to obtain necessary and sufficient conditions for the existence of p-periodic solutions of (5.1). We also give an application of this result to the computation of the approximate curves of transition from stability to instability in the Mathieu equation. We indicate also how the method used to obtain these approximations is related to a general scheme for obtaining the characteristic multipliers of linear periodic systems containing a small parameter.

To begin, we recall an elementary but important result on properties of linear operators in $\mathbb{R}^d$. Let the norm in $\mathbb{R}^d$ be the one induced by the inner product $(y, x) = y^*x$. Also, for any $d \times d$ matrix C, let $\mathcal{N}(C)$ (resp. $\mathcal{R}(C)$) denote the null space (resp. range) of C.

Lemma 7.1. *(The Fredholm Alternative for Matrices) For any $d \times d$ matrix C, we have*

$$\mathcal{N}(C^*) \oplus \mathcal{R}(C) = \mathbb{R}^d;$$

that is, the equation $Cx = b$ has a solution if and only if $\eta^*b = 0$ for all solutions η of the equation $C^*\eta = 0$.

If $\zeta^1, \zeta^2, \dots, \zeta^k$ is a basis for $\mathcal{N}(C)$, if $\eta^*b = 0$ for all η satisfying $C^*\eta = 0$ and ζ^0 is a solution of $Cx = b$, then the general solution of $Cx = b$ is given by $x = \zeta^0 + \Sigma_{j=1}^k c_j \zeta^j$, where $c_1, \dots, c_k$ are arbitrary constants.

Proof. For the first statement, we need only to observe that $(\eta, Cx) = 0$ for all $x \in \mathbb{R}^d$ is equivalent to $(C^*\eta, x) = 0$ for all $x \in \mathbb{R}^d$, which is equivalent to $C^*\eta = 0$. The last statement is obvious.

From Lemma 7.1, it follows that, if $\mathcal{R}(C) \neq \mathbb{R}^d$, then $\mathcal{N}(C) \neq \emptyset$. In fact, dim $\mathcal{N}(C) = \mathcal{N}(C^*)$ (prove this). Furthermore, $\mathcal{R}(C) = \mathbb{R}^d$ is equivalent to the statement: for each $b \in \mathbb{R}^d$, there is a unique soluton of the equation $Cx = b$. The work alternative is used to describe the lemma because either, for each $b \in \mathbb{R}^d$, there is a unique solution of the equation $Cx = b$ or there is a nontrivial vector x such that $Cx = 0$.

Theorem 7.1. *(The Fredholm Alternative for (5.1)) If (5.1) is a p-periodic system, then there is a p-periodic solution of (5.1) if and only if*

$$\int_0^p z^*(s)g(s)\,ds = 0 \tag{7.1}$$

for all p-periodic solutions of the adjoint equation (1.4).

If $\varphi^1, \dots, \varphi^k$ is a basis for the p-periodic solutions of (1.1), if (7.1) is satisfied for all p-periodic solutions of (1.4) and φ^0 is a p-periodic of (5.1), then the general p-periodic solution of (5.1) is given by $\varphi = \varphi^0 + \Sigma_{j=1}^k c_j \varphi^j$, where $c_1, \dots, c_k$ are arbitrary constants.

Proof. If $X(t)$, $X(0) = I$, is a fundamental matrix solution of the homogeneous equation (1.1) and $\psi(t)$ is a solution of (5.1), then the variation of constants formula implies that

$$\psi(t) = X(t)\psi(0) + \int_0^t X(t)X^{-1}(s)g(s)\,ds\,. \tag{7.2}$$

Since (5.1) is periodic and the solution of the initial value problem is unique, the function ψ will be a p-periodic solution of (5.1) if and only if $\psi(p) = \psi(0)$. Evaluating the above expression for ψ at $t = p$, we deduce that requiring that $\psi(p) = \psi(0)$ is equivalent to requiring that $\psi(0)$ be a solution of the equation

$$(X^{-1}(p) - I)\psi(0) = \int_0^p X^{-1}(s)g(s)\,ds\,. \tag{7.3}$$

A basic result in linear algebra asserts that this equation has a solution if and only if

$$\eta^* \int_0^p X^{-1}(s)g(s) = 0 \tag{7.4}$$

for all d-vectors η satisfying

$$(X^{-1}(p) - I)^*\eta = 0\,. \tag{7.5}$$

Since $Z(t) = (X^{-1}(t))^*$ is a fundamental matrix solution of the adjoint equation (1.4), we see that (7.4) is equivalent to the following:

$$0 = \int_0^p \eta^* X^{-1}(s)g(s)\,ds = \int_0^p ((X^{-1}(s))^*\eta)^* g(s)\,ds = \int_0^p (Z(s)\eta)^* g(s)\,ds\,.$$

The relation (7.5) is equivalent to $Z(p)\eta = \eta$ and so those η that satisfy (7.5) coincide with the initial data of the p-periodic solutions of the adjoint equation (1.4). This completes the proof of the first part of the theorem. The last part is obvious.

Corollary 7.1. *If (5.1) is a p-periodic system and the only p-periodic solution of (1.1) is the zero solution, then there is a unique p-periodic solution $\mathcal{K}g$ of (5.1) which is linear in g and there is a constant $c > 0$ such that, for $0 \le t \le p$,*

$$|(\mathcal{K}g)(t)| \le c \sup_{0\le s\le p} |g(s)|\,. \tag{7.6}$$

Proof. The hypothesis implies that (7.3) has a unique solution $\psi(0)(g)$ and it is easy to verify that $|\psi(0)(g)| \le c \sup_{0\le s\le p} |g(s)|$ for some constant c. This implies there is a unique p-periodic solution of (5.1) and it is given by (7.3) with $\psi(0) = \psi(0)(g)$. The estimate (7.6) is clear.

Theorem 7.2. *If (5.1) is a p-periodic system and there is a p-periodic solution z of the adjoint equation (1.4) such that* $\int_0^p z^*(s)g(s)ds \neq 0$*, then every solution of (5.1) is unbounded on* $[0, \infty)$.

Exercise 7.1. Prove Theorem 7.2.

Exercise 7.2. For a second order scalar differential equation, the Fredholm alternative can be stated without going to a system of two equations. In fact, if $a(t)$, $g(t)$ are p-periodic continuous functions, show that the equation $\ddot{u} + a(t)u = g(t)$ has a p-periodic solution if and only if $\int_0^p v(t)g(t)\,dt = 0$ for all p-periodic solutions $v(t)$ of the equation $\ddot{v} + a(t)v = 0$.

Exercise 7.3. Suppose that $\omega > 0$ is a given constant and show that the equation $\ddot{u} + u = \cos \omega t$ has a periodic solution if and only if $\omega \neq 1$.

Exercise 7.4. Show that every solution of (5.1) has period 2π if

$$A(t) = A = \frac{1}{2}\begin{bmatrix} -1 & 2 & 1 \\ -1 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad g(t) = \sin t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Example 7.1. (*The Mathieu Equation*) In Exercise 4.3, we have seen that the Mathieu Equation (4.5) has the following property: If $\delta \neq n^2$, n an integer, then there is a $\beta_0 = \beta_0(\delta) > 0$ such that the zero solution is uniformly stable if $\beta < \beta_0$. We now want to discuss what happens when $\delta = n^2$, n an integer. If $M(\delta, \beta)$ is the monodromy matrix of (4.5), then we know that it is analytic in (δ, β). Furthermore, the stability properties of the solutions are determined from Tr $M(\delta, \beta)$ (see Example 4.1). If $|\mathrm{Tr}\, M(\delta, \beta)| > 2$, the zero solution is unstable and if $|\mathrm{Tr}\, M(\delta, \beta)| < 2$, then the zero solution is uniformly stable. As a consequence, the only possibility of transition from uniform stability to instability of the zero solution is when $|\mathrm{Tr}\, M(\delta, \beta)| = 2$, which coincides with the points (δ, β) for which (4.5) has π-periodic or 2π-periodic solutions. As a consequence, it is important to determine the points where $|\mathrm{Tr}\, M(\delta, \beta)| = 2$. These curves can and have been determined numerically for a large part of the (δ, β)-space. We want to show how to use the Fredholm alternative to determine some of these for β small.

Figure 7.1.

For $\beta = 0$, it is simple to see that the graph of Tr $M(\delta, 0)$ has the form shown in Figure 7.1. For $\delta < 0$, $TrM(\delta, 0) > 2$ and, for $\delta > 0$, $|\mathrm{Tr}\, M(\delta, 0)| \leq 2$. The function Tr $M(\delta, 0) - 2$ has a simple zero at $\delta = 0$ and a double zero at $\delta = (2k)^2$, $k = 1, 2 \ldots$. At these points, both Floquet multipliers are $+1$ with only one π-periodic solution at $\delta = 0$ and a basis of such solutions at the other values of δ. The function Tr $M(\delta, 0) + 2$ has a double zero at $\delta = (2k-1)^2$, $k = 1, 2 \ldots$, both characteristic multipliers are -1 and there is a basis of 2π-periodic solutions.

If β is small, then then the graph of Tr $M(\delta, \beta)$ will have the same form as in Figure 7.1. There will be a simple zero $\delta_0^*(\beta)$, $\delta_0^*(0) = 0$, of the function Tr $M(\delta, \beta) - 2$. On the other hand, it is not clear that this function will have zeros near the points $((2k)^2, 0)$, $k \geq 1$, or that the function Tr $M(\delta, \beta) + 2$ will have zeros near the points $((2k-1)^2, 0)$, $k \geq 1$. If there were simple zeros $\delta_{k,1}^*(\beta)$, $\delta_{k,2}^*(\beta)$ near each of these points, then the graph of Tr $M(\delta, \beta)$ will have the form shown in Figure 7.2 for β small. The shaded regions correspond to regions of instability of the zero solution of (4.5).

Figure 7.2.

In the (δ, β)-plane, the graphs of the functions $\delta_0^*(\beta)$, $\delta_{k,1}^*(\beta)$, $\delta_{k,2}^*(\beta)$ represent the curves of transition from uniform stability to instability of the zero solution of (4.5) (see Figure 7.3).

Figure 7.3.

We now compute approximate expressions for some of these transition curves. Since we expect two branches of in the (δ, β)-plane near the points $(n^2, 0)$ (if they exist), we should try to determine these branches by finding δ as a function of β. Assume that $u(t, \beta)$ is a periodic solution of (4.4) with $\delta = \delta(\beta)$, $\delta(0) = n^2$, n an integer. The period of $u(t, \beta)$ is to be π if n is either even or zero and $2pi$ if n is odd.

Also, let us suppose that $u(t,\,\beta)$, $\delta(\beta)$ have power series expansions of the form

$$
\begin{aligned}
u(t,\,\beta) &= u_0(t) + u_1(t)\beta + u_2(t)\frac{\beta^2}{2} + \ldots \\
\delta(\beta) &= n^2 + \delta_1\beta + \delta_2\frac{\beta^2}{2} + \ldots\,,
\end{aligned}
\tag{7.7}
$$

near $\beta = 0$, where each $u_j(t)$, $j \geq 1$ is periodic. In order to determine the $u_j(t)$, $j \geq 1$ in a unique way, we require that u_k, $k \geq 1$, be orthogonal to u_0, $\dot u_0$; that is, $\int_0^p u_0 u_k = 0$, $\int_0^p \dot u_0 u_k = 0$, $k \geq 1$. We could use other ways to uniquely define these functions; for example, the initial values at $t = 0$, but we prefer the more geometric one using orthogonality.

If we substitute the expressions (7.7) into (4.5) and equate coefficients in like powers of β, then we obtain

$$
\begin{aligned}
\ddot u_0 + n^2 u_0 &= 0 \\
\ddot u_1 + n^2 u_1 &= -(\delta_1 + \cos\,2t)u_0 \\
\ddot u_2 + n^2 u_2 &= -[(\delta_1 + \cos\,2t)u_1 + \delta_2 u_0]\,.
\end{aligned}
\tag{7.8}
$$

We will apply the Fredholm Alternative (Theorem 7.1) in the form stated in Exercise 7.2. Let us first consider the case when $n = 0$. In this case, we are seeking the function $u(t,\,\beta)$ as a π-periodic function. The function $u_0(t) = 1$ is the only π-periodic solution of $\ddot u = 0$. From Exercise 7.2, the second equation in (7.8) for $n = 0$ has a π-periodic solution if and only if the function $\delta_1 + \cos\,t$ is orthogonal to the constant function 1; that is, $\delta_1 = -\frac{1}{\pi}\int_0^\pi \cos\,2t\,dt = 0$. With this choice of $\delta_1 = 0$, the second equation in (7.8) has a π-periodic solution. Our requirement that it be orthogonal to $u_0(t) = 1$, $\dot u_0(t) = 0$, we see that $u_1(t)$ must be given by $u_1(t) = \frac{1}{4}\cos 2t$. With this choice of $u_1(t)$, the last equation in (7.8) for $n = 0$ has a π-periodic solution if and only if the function $\frac{1}{4}\cos^2 t + \delta_2$ is orthogonal to the constant function 1; that is, $\delta_2 = -\frac{1}{4\pi}\int_0^\pi \cos^2\,2t\,dt = -\frac{1}{8}$. This means that the curve of transition from instability to stability in the $(\delta,\,\beta)$- plane near (0, 0 is given by $\delta(\beta) = -\frac{1}{8}\beta^2 + \cdots$.

Let us now suppose that $n = 1$. In this case, we want the function $u(t,\,\beta)$ to be 2π-periodic. There are two independent possibilities for u_0; namely, $u_0(t) = \cos\,t$, $u_0(t) = \sin\,t$. If $u_0(t) = \cos\,t$, then u_1 satisfies the differential equation

$$
\ddot u_1 + u_1 = -(\delta_1 + \frac{1}{2})\cos\,t - \frac{1}{2}\cos\,3t\,.
$$

From Exercise 7.2, the Fredholm alternative implies that the right hand side of this equation must be orthogonal to all 2π-periodic solutions of the equation $\ddot u + n^2 u = 0$. A basis for these solutions is $\sin\,t$ and $\cos\,t$. Since the right hand side must be orthogonal to $\sin\,t$, we deduce that there is a 2π-periodic solution u_1 if and only if

$$
\int_0^{2\pi} [(\delta_1 + \frac{1}{2})\cos\,t + \frac{1}{2}\cos\,3t]\cos\,t\,dt = 0
$$

and so $\delta_1 = -\frac{1}{2}$. With this choice of δ_1 and the requirement that $u_1(t)$ is orthogonal to both $\cos t$ and $-\sin t$, we see that $u_1(t) = \frac{1}{16}\cos 3t$. Continuing this process to the next step in the approximation, we deduce that $\delta_2 = -\frac{1}{32}$. Therefore, the curve of transition from stability to instability is given by $\delta(\beta) = 1 - \frac{1}{2}\beta - \frac{1}{32}\beta^2 + \cdots$.

If we take $u_0(t) = \sin t$ and carry out the same type of computations, then we obtain another curve of transition given by $\delta(\beta) = 1 + \frac{1}{2}\beta - \frac{1}{32}\beta^2 + \cdots$. Since these two curves are distinct, it follows that there is a region of instability near (1, 0) as s shown in Figure 7.3.

Exercise 7.5. In (7.7), take $n = 1$ and $u_0(t) = a\cos t + b\sin t$ and carry out the above computations requiring that $a^2 + b^2 \neq 0$. Show that δ_1 must be an eigenvalue of a special 2×2 matrix.

Exercise 7.6. Near the point (4, 0), show that the curves of transition from stability to instability in the (δ, β)-plane are given approximately by

$$\delta = 4 \pm \frac{5}{48}\beta^2 + \cdots .$$

It is possible to give asymptotic formulas for the curves of transition from stability to instability for Mathieu's equation for any value of n. In fact, we state without proof the following result. (See Hale, Contributions to the Theory of Nonlinear Oscillations **5** (1960), 55-89.)

Theorem 7.3. *For Mathieu's Equation, the curves of transition from stability to instability near* $(n^2, 0)$, $n \geq 1$, *in the* (δ, β)*-plane are given by* $\delta(\beta) = n^2 \pm \delta_n\beta^n + \cdots$, *where* $\delta_n \neq 0$.

Theorem 7.3 implies that there is going to be a region of instability near each of the points $(n^2, 0)$ and the graph of Tr $M(\delta, \beta)$ near $\beta = 0$ has the form shown in Figure 7.2.

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