2.5. Nonhomogeneous Linear Systems.

If $A \in C^0(\mathbb{R}, \mathbb{R}^{d\times d})$ (or $A \in C^0(\mathbb{R}, \mathbb{C}^{d\times d})$) is a $d \times d$ matrix function and $g \in C(\mathbb{R}, \mathbb{R}^d)$ (or $g \in C(\mathbb{R}, \mathbb{C}^d)$), we consider the linear equation

$$\dot{x} = A(t)x + g(t) \tag{5.1}$$

and refer to it as a *linear nonhomogeneous equation.* For any $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$, Theorem 1.3.1 implies that there is a unique solution $x(t, \tau, \xi)$ through (τ, ξ) which is continuous in its arguments and Exercise 1.4.2 ensures that this solution exists for all $t \in (-\infty, \infty)$. We have the following representation of the solution.

Theorem 5.1. *(Variation of Constants Formula) If $X(t)$ is a fundamental matrix solution of (1.1), then the solution $x(t, t, \xi)$ of (5.1) is given by*

$$x(t, \tau, \xi) = X(t)X^{-1}(\tau)\xi + \int_\tau^t X(t)X^{-1}(s)g(s)\, ds\,. \tag{5.2}$$

In particular, if A is a constant matrix, then

$$x(t, \tau, \xi) = e^{A(t-\tau)}\xi + \int_\tau^t e^{A(t-s)}g(s)\, ds\,. \tag{5.3}$$

Proof. If we make the transformation of variables $x(t, \tau, \xi) = X(t)c(t)$ in (5.1), then a simple computation shows that $\dot{c} = X^{-1}(t)g$ and so

$$\begin{aligned} c(t) &= c(\tau) + \int_\tau^t X^{-1}(s)g(s)\, ds \\ &= X^{-1}(\tau)\xi + \int_\tau^t X^{-1}(s)g(s)\, ds\,. \end{aligned}$$

This proves (5.2). Relation (5.3) is clear from the definition of e^{At}.

Exercise 5.1. (*Nonlinear Variation of Constants Formula*) There is an interesting generalization of (5.2) to nonlinear equations. Consider the equation

$$\dot{x} = f(t, x) + g(t, x)\,, \tag{5.4}$$

where $f, g \in C^r(\mathbb{R}, \mathbb{R}^d)$, $r \geq 1$. Let $\varphi(t, \tau, \xi)$ be the solution of (5.4) satisfying $\varphi(\tau, \tau, \xi) = \xi$ and let $\psi(t, \tau, \xi)$ be the solution of the "unperturbed" nonlinear equation

$$\dot{x} = f(t, x) \tag{5.5}$$

satisfying $\psi(\tau, \tau, \xi) = \xi$. Prove that

$$\varphi(t, \tau, \xi) = \psi(t, \tau, \xi) + \int_\tau^t \frac{\partial\psi(t, s, \varphi(s, \tau, \xi))}{\partial\xi} g(s, \varphi(s, \tau, \xi))\, ds\,. \tag{5.6}$$

Hint. We know that (see Formula (1.3.4))

$$\frac{\partial\psi(t, \tau, \xi)}{\partial\tau} = -\frac{\partial\psi(t, \tau, \xi)}{\partial\xi} f(\tau, \xi)\,.$$

Show that

$$\frac{d\psi(t, s, \varphi(s, \tau, \xi))}{ds} = \frac{\partial\psi(t, s, \varphi(s, \tau, \xi))}{\partial\xi} g(s, \varphi(s, \tau, \xi))$$

and then integrate from 0 to t.

It is important to know that the solutions of (5.1) depend continuously upon the matrix A and function h. In the uniform topology, this is a consequence of the uniform contraction principle. We give an independent proof of this fact which shows also that there is continuous dependence in a weaker topology.

Theorem 5.2. *Let A, $B \in C(\mathbb{R}, \mathbb{R}^{d\times d})$ (or A, $B \in C(\mathbb{R}, \mathbb{C}^{d\times d})$) be $d \times d$ matrix functions and let g, $h \in C(\mathbb{R}, \mathbb{R}^d)$ (or g, $h \in C(\mathbb{R}, \mathbb{C}^d)$). If $x(t)$ is a solution of the equation (5.1) and $y(t)$ is a solution of*

$$\dot{y} = B(t)y + h(t) \tag{5.7}$$

then, for any $\tau \in \mathbb{R}$, we have

$$\begin{aligned} |x(t) - y(t)| &\leq e^{\int_\tau^t |A(s)|\, ds} |x(\tau) - y(\tau)| \\ &\quad + \int_\tau^t \left(e^{\int_s^t |A(u)|\, du}\right)[|A(s) - B(s)|\, |y(s)| + |g(s) - h(s)|]\, ds\,. \end{aligned} \tag{5.8}$$

Proof. If we let $z = x - y$, then $\dot{z} = Az + (A - B)y + g - h$. This implies that

$$|z(t)| \leq |z(\tau)| + \int_\tau^t [|A(s)|\, |z(s)| + |A(s) - B(s)|\, |y(s)| + |g(s) - h(s)|]\, ds\,.$$

An application of the Generalized Gronwall Inequality yields the result stated in the theorem.

Exercise 5.2. Let $X_A(t)$, $X_A(0) = I$, be a fundamental matrix solution of (1.1). Prove that $X_A(t)$ is continuous in the matrix A in the sense that, for any $\epsilon > 0$ and any t_0, there is a $\delta > 0$ such that $\int_0^{t_0} |A(s) - B(s)|\, ds < \delta$ implies that, for $0 \leq t \leq t_0$, we have $|X_A(t) - X_B(t)| < \epsilon$.

Exercise 5.3. Let $p > 0$ be a given constant and let X_p be the Banach space of $d \times d$ continuous matrix functions $B(t)$, $t \in \mathbb{R}$, $B(t+p) = B(t)$ for all t with the *norm topology*, $\|B(\cdot)\| = \max_{t \in \mathbb{R}} \|B(t)\|$. Prove that the Floquet multipliers of (1.1) are continuous functions of the matrix function $A(\cdot) \in X_p$.

Exercise 5.4. Suppose that X_p is defined as in Exercise 5.3 and that $A(\cdot) \in X_p$ has the property that the solutions of (1.1) approach zero as $t \to \infty$. Prove that there is a neighborhood U of $A(\cdot)$ in X_p such that, for any $B(\cdot) \in U$, the solutions of $\dot{x} = B(t)x$ approach zero as $t \to \infty$.

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