2.4. Linear Periodic Systems.

If $A \in C^0(\mathbb{R}, \mathbb{R}^{d\times d})$ (or $A \in C^0(\mathbb{R}, \mathbb{C}^{d\times d})$) is a $d \times d$ matrix function and there is a constant $p > 0$ such that $A(t + p) = A(t)$ for all t, then we say that the differential equation (1.1) is a *periodic system* or a *p-periodic system* if we want to emphasize the period. We remark that we do not assume that p is the minimal period. Our objective is to give a complete characterization of the general structure of the solutions of a periodic system (1.1).

If $X(t)$ is a $d \times d$ matrix solution of a p-periodic system (1.1), then $X(t+p)$ also is a solution. If $X(t)$ is a fundamental matrix solution, then there is $d \times d$ constant matrix M such that $X(t+p) = X(t)M$ for all t. Furthermore, M is nonsingular. The matrix M is called a *monodromy matrix* for (1.1) and the eigenvalues ρ are called the *Floquet multipliers* of (1.1). Since each Floquet multiplier ρ is different from zero, there is complex number λ, called a *Floquet exponent*, such that $\rho = e^{\lambda p}$. We remark that the Floquet multipliers and the real parts of the characteristic exponents are uniquely defined, but the imaginary parts of the characteristic exponents are not. The first question that must be asked is: do the Floquet multipliers depend upon the fundamental solution $X(t)$? If $Y(t)$ is another fundamental solution of (1.1), then there is a nonsingular matrix D such that $Y(t) = X(t)D$ and so $Y(t+p) = X(t+p)D = X(t)MD = Y(t)D^{-1}MD$ and the monodromy matrix for $Y(t)$ is $D^{-1}MD$. Since this matrix is similar to M, the eigenvalues are the same and the Floquet multipliers do not depend upon the choice of the fundamental matrix. As a consequence of this fact, we can always take $X(0) = I$ in defining the monodromy matrix.

We remark that the Floquet multipliers do depend upon the period p. In fact, consider the example $\dot{x} = (-1 + \cos 2\pi t)x$, for which the solution is $x(0)\exp[-t + [2\pi]^{-1}\sin 2\pi t]$. If we consider the period of the coefficients in the equation to be 1, then the Floquet multiplier is $\exp(-1)$. If we consider the period of the coefficients to be 2, then the Floquet multiplier is $\exp(-2)$.

Let us recall that, for an autonomous linear equation $\dot{x} = Ax$, if λ is an eigenvalue of A, then there is a nonzero vector ξ such that $e^{\lambda t}\xi$ is a solution. For a p-periodic system (1.1), the Floquet exponents play the role of eigenvalues in the constant coefficient case. Of course, we cannot expect to have a solution as simple as a constant vector times an exponential. However, we have the following result.

Lemma 4.1. *A complex number $\rho = e^{\lambda p}$ is a Floquet multiplier of a p-periodic system (1.1) if and only if there is a nontrivial solution of (1.1) of the form $e^{\lambda t}q(t)$, where q is p-periodic. In particular, there is a periodic solution of period p (resp. 2p and not p) if and only if there is a multiplier $= +1$ (resp. -1).*

Proof. If $e^{\lambda t}q(t)$, $q(t + p) = q(t) \neq 0$, is a solution of (1.1) and we choose a fundamental matrix solution $X(t) = [e^{\lambda t}q(t), q_2(t), \dots, q_d(t)]$, then $X(t)\mathrm{col}[1, 0, \dots, 0] = e^{\lambda t}q(t)$ and $M\mathrm{col}[1, 0, \dots, 0] = e^{\lambda p}\mathrm{col}[1, 0, \dots, 0]$, where M is the monodromy matrix. This proves that $\rho = e^{\lambda p}$ is a Floquet multiplier of (1.1)

Conversely, if $\rho = e^{\lambda p}$ is a Floquet multiplier of (1.1); that is, an eigenvalue of the monodromy matrix M of a fundamental matrix solution $X(t)$ of (1.1), and $M\xi = e^{\lambda p}\xi$ with $\xi \neq 0$, then it is easy to see that $q(t) \equiv X(t)\xi e^{-\lambda t} = q(t+p) \neq 0$ for all t and $q(t)e^{\lambda t}$ is a nontrivial solution of (1.1).

In the statement of Lemma 4.1, we singled out the periodic solutions of period p and $2p$ because of the special role that they play in some of our later examples. If k is a positive integer, then it is clear that there is a periodic solution of period kp and not of period jp for any integer $j < k$ if and only if there is a Floquet multiplier $e^{i\omega p}$ which is a k^{th} root of unity; that is, ω is real and $e^{ik\omega p} = 1$, $e^{ij\omega p} \neq 1$ for $j < k$.

To describe the solutions of (1.1) that are not periodic, it is convenient to introduce some terminology. A set $\{\omega_1, \omega_2, \ldots, \omega_s\}$ of real numbers is said to be *rationally independent* if $\Sigma_{j=1}^{s} r_j\omega_j = 0$ for rational numbers r_j, $j = 1, \ldots, s$ implies each $r_j = 0$. A continuous function h is said to be *quasiperiodic* with basic frequencies $\omega_1, \omega_2, \ldots, \omega_s$ if the set $\{\omega_1, \omega_2, \ldots, \omega_s\}$ is rationally independent and there is a continuous function $H(\theta_1, \theta_2, \ldots, \theta_s)$, periodic in θ_j of minimal period $\frac{2\pi}{\omega}$, $j = 1, \ldots, s$, such that $h(t) = H(t, t, \ldots, t)$ for all $t \in \mathbb{R}$. The function $\sin t + \sin \pi t$ is quasiperiodic with basic frequencies $1, \pi$.

If $e^{i\omega p}, \omega$ real, is a Floquet multiplier and not a root of unity, then the nontrivial solution $e^{i\omega t}q(t)$ is a quasiperiodic function with basic frequencies ω, $\frac{2\pi}{p}$. If $\rho = e^{\alpha + i\omega t}$ is a Floquet multiplier and $e^{i\omega p}$ is not a root of unity, then the solution $e^{\alpha + i\omega t}q(t)$ is the product of a quasiperiodic function and an exponential function.

Lemma 4.2. *If ρ_j, $j = 1, 2, \ldots, d$, are the Floquet multipliers of the p-periodic system (1.1), then*

$$\prod_{j=1}^{d} \rho_j = e^{\int_0^p \operatorname{Tr} A(s)\, ds}. \tag{4.1}$$

Proof. If M is the monodromy matrix of the fundamental solution $X(t)$, $X(0) = I$ of (1.1), then Liouville's Theorem 2.2 implies (4.1).

At first glance it might appear that linear periodic systems share the same simplicity as autonomous linear systems. However, there is a very important difference - the Floquet multipliers are obtained from the solutions of the differential equation and there is no apparent connection with the coefficients of the matrix in the differential equation. The following exercise illustrates this fact.

Exercise 4.1. Consider the 2π-periodic system (1.1) with

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$$

Show that the eigenvalues of $A(t)$ are $\frac{1}{4}[-1 \pm i\sqrt{7}]$ and thus have real parts negative. Verify that the unbounded function col $(-\cos t, \sin t)e^{t/2}$ is a solution of the equation. Use Lemma 4.2 to show that the multipliers are e^{π} and $e^{-2\pi}$.

To obtain more information about the structure of a fundamental matrix, we need the following exponential representation for a nonsingular matrix.

Lemma 4.3. *If M is a nonsingular $d \times d$ matrix, then there is $d \times d$ matrix B such that $M = e^B$.*

Proof. We first note that, if $M = e^B$ and P is a nonsingular matrix, then $PMP^{-1} = e^{PBP^{-1}}$. Thus, without loss of generality, we may assume that M is in Jordan Canonical Form, $M = \operatorname{diag}(M_1, \ldots, M_s)$, where each M_j has the form: $M_j = \lambda_j + N_j = \lambda_j(I + \frac{N_j}{\lambda_j})$, where λ_j is a complex number and N_j is nilpotent; that is, there is an integer d_j such that, for $k \geq d_j$, we have $N_j^k = 0$. Using the power series for $\log(1+t)$ near $t = 0$ as motivation, if we define

$$B_j = (\log \lambda_j)I - \Sigma_{k=1}^{d_j-1} \frac{(-N_j)^k}{k\lambda_j^k} \equiv (\log \lambda_j)I + S_j,$$

then

$$\begin{aligned} e^{B_j} &= e^{\log \lambda_j I} e^{S_j} \\ &= \lambda_j \Sigma_{i=0}^{\infty} \frac{1}{i!}\big[- \Sigma_{k=1}^{\infty} \frac{(-N_j)^k}{k\lambda_j^k}\big]^i \\ &= \lambda_j (I + \frac{N_j}{\lambda_j}) = M_j \,. \end{aligned}$$

If we define $B = \operatorname{diag}(B_1, \ldots, B_s)$, then $e^B = M$ and the lemma is proved.

Exercise 4.2. Suppose that M is a nonsingular matrix. If there is no negative eigenvalue of M, show that there is a *real* matrix B such that $M = e^B$. For a general nonsingular matrix M, show that there is *real* matrix B such that $M^2 = e^B$.

Theorem 4.1. *(Floquet representation) If $X(t)$ is a fundamental matrix for the p-periodic system (1.1), then there exist a $d \times d$ constant matrix B and nonsingular $d \times d$ p-periodic matrix $P(t)$ such that*

$$X(t) = P(t)e^{Bt} \,. \tag{4.2}$$

In addition, there are a real matrix B and a $2p$-periodic nonsingular matrix $P(t)$ such that (4.2) holds.

Proof. If M is the monodromy matrix for $X(t)$, then Lemma 4.1 implies that there is a matrix B such that $M = e^{Bp}$. If we define $P(t) = X(t)e^{-Bt}$, then

$$P(t+p) = X(t+p)e^{-B(t+p)} = X(t)e^{Bp}e^{-B(t+p)} = P(t) \,,$$

which proves (4.2).

If we choose the real matrix B so that $M^2 = e^{B2p}$, then $P(t)$ is real and

$$P(t+2p) = X(t+2p)e^{-B(t+2p)} = X(t+p)Me^{-B(t+2p)} = M^2 e^{-B(t+2p)} = P(t) \,.$$

This completes the proof of the theorem.

Corollary 4.1. *For the p-periodic system (1.1), there is a nonsingular p-periodic (real nonsingular 2p-periodic) transformation of variables taking it into an equation with constant coefficients.*

Proof. If $P(t)$, B are defined by (4.2) and $x = P(t)y$, then the equation for y is

$$\dot{y} = P^{-1}(AP - \dot{P})y\,.$$

Since $P = Xe^{-Bt}$, it follows that $\dot{P} = AP - PB$ and so $\dot{y} = By$.

For any $\xi \in \mathbb{R}^d$, we know from Section 1.5 that $e^{Bt}\xi$ is the sum of exponential functions with polynomial coefficients. From the Floquet representation, it follows that any solution of the p-periodic system (1.1) is the sum of functions of the form $e^{\lambda t}q(t)$, where $q(t)$ is a polynomial in t with coefficients which are p-periodic in t. The numbers λ are the Floquet exponents.

As an immediate consequence of the Floquet representation and Theorem 1.7.2, we obtain the following result on stability.

Theorem 4.2. *For the p-periodic system (1.1), if M is a monodromy matrix, then the solution $x = 0$ is*

(i) uniformly stable if and only if each $\rho \in \sigma(M)$ satisfies $|\rho| \le 1$ and the ones with $|\rho| = 1$ have simple elementary divisors;

(ii) uniformly asymptotically stable if and only if each $\rho \in \sigma(M)$ satisfies $|\rho| < 1$.

Example 4.1. (*Hill's Equation*) If δ is a constant and $b(t)$ is a π-periodic function, the equation

$$\ddot{u} + [\delta + b(t)]u = 0 \tag{4.3}$$

is referred to as *Hill's Equation.* If we write this equation as a system (1.1) for the vector $x = \text{col}\,(u, \dot{u})$, then the matrix $A(t)$ is given by

$$A(t) = \begin{bmatrix} 0 & 1 \\ -[\delta + b(t)] & 0 \end{bmatrix}.$$

If M is the monodromy matrix for the fundamental solution $X(t)$, $X(0) = I$, then the Floquet multipliers ρ_1, ρ_2 of (4.3) are the solutions of the equation

$$\det\,[M - \rho I] = \rho^2 + (\mathrm{Tr}\,M)\rho + 1 = 0\,, \tag{4.4}$$

since $\det\, M = 1$ by (4.1).

From Theorem 4.2 and (4.4), we can determine the stability properties of the zero solution of (4.3) from the magnitude of Tr M. In fact, the only possibilities are the following:

(i) If $|\mathrm{Tr}\, M| < 2$, then the Floquet multipliers are complex and simple: $|\rho_1| = |\rho_2| = 1$, $\rho_1 = \bar{\rho}_2$, and the zero solution of (4.3) is uniformly stable.

(ii) If $|\mathrm{Tr}\, M| > 2$, then either $0 < \rho_1 < 1 < \rho_2$ or $\rho_2 < -1 < \rho_1 < 0$ and the zero solution of (4.3) is unstable.

(iii) If $|\mathrm{Tr}\, M| = 2$, then either $\rho_1 = \rho_2 = 1$ or $\rho_1 = \rho_2 = -1$. In the first situation, there must be a π-periodic solution of (4.3) and, in the latter situation, there must be a 2π-periodic solution of (4.3). The solutions of (4.3) are either all periodic or periodic functions times a linear function of t depending upon whether or not the monodromy matrix is diagonalizable.

The computation of Tr M is difficult and generally it only can be done by some approximation procedure. For some important cases that appear in the applications, the special functions defined by (4.3) have been tabulated. If $b(t) = \delta + \beta \cos 2t$, the equation (4.3) is called *Mathieu's Equation* and the zones of stability in the (δ, β) plane as well as fundamental matrix solutions have been tabulated. The only possible transition regions from stability to instability are those values of (δ, β) for which the equation has a periodic solution of period π or 2π. After we have developed more analytic machinery, we will return to this example and obtain a part of these transition curves.

Exercise 4.3. For the Mathieu equation,

$$\ddot{u} + (\delta + \beta \cos 2t)u = 0\,, \tag{4.5}$$

show that, if $\delta \neq n^2$, where $n \geq 0$ is an integer, then there is a $\beta_0 = \beta_0(\delta) > 0$ such that, for $0 \leq \beta \leq \beta_0$, the zero solution of (4.5) is uniformly stable.
Hint. For $\beta = 0$, consider the linear equation $\ddot{u} + \delta u = 0$ as a π-periodic equation and show that the Floquet multipliers are nonreal and have modulus one if $\delta \neq n^2$, where $n \geq 0$ is an integer. Now use the fact that the Floquet multipliers of (4.5) are continuous functions of β. This latter fact is a consequence of the continuous dependence of solutions on parameters.

Exercise 4.4. (Arnol'd, p. 205) The Floquet theory for periodic systems is valid for coefficients which are integrable in t; in particular, for piecewise continuous functions. Suppose that ϵ is a small parameter, ω is a constant, $b(t) = \omega + \epsilon$, $0 \leq t < \pi$, $= \omega - \epsilon$, $\pi \leq t < 2\pi$, and consider the equation

$$\ddot{u} + b^2(t)u = 0.$$

Show that the approximate formulas for the curves of transition from stability to instability for this equation for ϵ small are given by

$$\omega = k \pm \frac{\epsilon^2}{k} + o(\epsilon^2)$$
$$\omega = k + \frac{1}{2} \pm \frac{\epsilon}{\pi(k + \frac{1}{2})} + o(\epsilon)\,,$$

where k is a positive integer.
Hint. Let $\omega_1 = \omega + \epsilon$, $\omega_2 = \omega - \epsilon$, $c_j = \cos\omega_j$, $s_j = \sin\omega_j$, $j = 1, 2$, and show that the fundamental matrix solution $X(t)$, $X(0) = I$, of the corresponding system satisfies

$$\frac{1}{2} Tr X(2\pi) = c_1 c_2 - (1+d) s_1 s_2\,,$$

where $1 + d = \frac{1}{2}(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1})$ and $d = \frac{2\epsilon^2}{\omega^2} + O(\epsilon^4)$. The transition from stability to instability is when $|Tr X(2\pi)| = 2$. Use trigonometric formulas to show that this is equivalent to either

$$\cos 2\pi\omega = 1 - \frac{d}{2+d}(1 - \cos 2\pi\epsilon)$$
$$\cos 2\pi\omega = -1 + \frac{d}{2+d}(1 + \cos 2\pi\epsilon)\,.$$

Solve these equations approximately.

æ