2.2. Liouville's Theorem.

We recall a few elementary facts from linear algebra. If $B = (b_{ij})$ is a $d \times d$ matrix, let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be the not necessarily distinct eigenvalues of B. Then we know that $\det B = \prod_{j=1}^{d} \lambda_j$ and $\mathrm{Tr}\ B = \Sigma_{j=1}^{d} b_{jj} = \Sigma_{j=1}^{d} \lambda_j$. Also, if A, B are $d \times d$ matrices, then $\det AB = \det A \det B$ and if, in addition, $\det B \neq 0$, then $\mathrm{Tr}\ A = \mathrm{Tr}\ B^{-1}AB$.

If $\xi_1, \ldots, \xi_d$ are given vectors in $\mathbb{R}^d$, then the *parallelepiped* with these edges is the set consisting of all points of the form $x_1\xi_1 + \cdots + x_d\xi_d$, $0 \leq x_j \leq 1$, $j = 1, \ldots, d$. The determinant of a matrix A is the oriented volume of the parallelepiped whose edges are given by the columns of A. In this section, we derive a formula in terms of the coefficients of the matrix $A(t)$ for $\det X(t)$ where $X(t)$ is a fundamental matrix solution of (1.1). For any set $B \subset \mathbb{R}^d$, we can define the image of this set under (1.1) by the relation $X(t)B = \{\, X(t)\xi : \xi \in B \,\}$. Using the formula for $\det X(t)$, we see how the volume of the set $X(t)B$ changes with t. If this determinant is < 1 (resp., > 1) for all t, then the volume is decreasing (resp., increasing). If $\det X(t) = 1$ for all t, the volume remains constant and we say that (1.1) is *volume preserving.*

We need first a lemma on matrices depending on a parameter.

Lemma 2.1. *If $A(\epsilon)$ is a C^1 $d \times d$ matrix function of ϵ for $\epsilon \in (-\epsilon_0, \epsilon_0)$, satisfying $A(0) = I$, $\frac{d}{d\epsilon}A(\epsilon)_{\epsilon=0} = A_0$, then*

$$\frac{d}{d\epsilon} \det\ A(\epsilon)_{\epsilon=0} = \mathrm{Tr}\, A_0 \,. \tag{2.1}$$

Proof. Since $A(0) = I$, we may change coordinates and assume that $A(\epsilon) = I + A_0\epsilon + o(\epsilon)$ and A_0 is in Jordan canonical form. If $1+\lambda_1(\epsilon), \ldots, 1+\lambda_d(\epsilon)$ are the eigenvalues of $A(\epsilon)$ and $\lambda_{10}, \ldots, \lambda_{d0}$ are the eigenvalues of A_0, then $\lambda_j(\epsilon) = \epsilon\lambda_{j0} + o(\epsilon)$ as $\epsilon \to 0$ for each $j = 1, \ldots, d$, and

$$\det\ A(\epsilon) = \prod_{j=1}^{d}(1 + \lambda_j(\epsilon)) = 1 + \epsilon\Sigma\lambda_{0j} + o(\epsilon) = 1 + \epsilon\,\mathrm{Tr}\, A_0 + o(\epsilon)$$

as $\epsilon \to 0$. This relation implies the conclusion in the lemma.

Theorem 2.1. *(Liouville's Theorem) If $A \in C(\mathbb{R}, \mathbb{R}^{d\times d})$ is a $d \times d$ matrix and $X(t)$ is a matrix solution of (1.1), then*

$$\det X(t) = \det X(\tau) e^{\int_\tau^t \mathrm{Tr}\, A(s)\, ds} . \tag{2.2}$$

Proof. If $\det X(\tau) = 0$, then (2.2) is clearly satisfied from Proposition 1.1. Therefore, we assume that $\det X(\tau) \neq 0$. Since $X(t)$ is a C^1-function, we have $\det X(t)$ is a C^1-function. For any $s \in \mathbb{R}$, since $X(t)$ is a matrix solution of (1.1), we know that

$$X(t)[X(s)]^{-1} = I + \int_s^t A(\tau)X(\tau)[X(s)]^{-1} d\tau.$$

If we let $\epsilon = t - s$, $B(\epsilon) = X(\epsilon + s)[X(s)]^{-1} \equiv X(t)[X(s)]^{-1}$, then $B(0) = I$, $\frac{d}{d\epsilon}B(\epsilon)_{\epsilon=0} = A(s)$ and Lemma 2.1 implies that $\frac{d}{d\epsilon}\det B(\epsilon)_{\epsilon=0} = \operatorname{Tr} A(s)$. Since the determinant of a product is the product of the determinants, we have

$$(\frac{d}{dt}\det X(t))\det[X(s)]^{-1} = \frac{d}{dt}\det(X(t)[X(s)]^{-1}).$$

Using this fact and the relation $\frac{d}{d\epsilon}\det B(\epsilon) = \frac{d}{dt}\det(X(t)[X(s)]^{-1})$, we deduce that

$$(\frac{d}{dt}\det X(t)_{t=s})\det[X(s)]^{-1} = \operatorname{Tr} A(s)\,,$$

or equivalently the function $u(s) = \det X(s)$ is a solution of the scalar equation $\dot{u} = \operatorname{Tr} A(s)u$, $u(\tau) = \det X(\tau)$. An integrating of this equation yields (2.2).

Example 2.1. (*Abel's formula*) Consider the second order equation

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0, \tag{2.3}$$

where p, q are continuous functions on $\mathbb{R}$. This equation is equivalent to a two dimensional system (1.1) with

$$A(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}.$$

For any two scalar functions u, v, the *Wronskian* $W(u, v, t)$ of u, v is defined to be

$$W(u, v, t) = \det\begin{bmatrix} u(t) & v(t) \\ \dot{u}(t) & \dot{v}(t) \end{bmatrix}.$$

If u, v are two solutions of the second order equation, then

$$X(t) = \begin{bmatrix} u(t) & v(t) \\ \dot{u}(t) & \dot{v}(t) \end{bmatrix}$$

is a matrix solution of the second order system. As a consequence of Liouville's Theorem, we conclude that

$$W(u, v, t) = W(u, v, \tau)e^{-\int_\tau^t p(s)\,ds}\,,$$

which is Abel's formula.

If we suppose that $p(t) = 0$ for all t, then (2.3) is area preserving. If $p(t) > 0$ for all t, then the area of any set is decreasing.

Exercise 2.1. Make an appropriate definition of a Wronskian and extend the previous example to an n^{th} order scalar equation.

From Liouville's Formula (2.2), if Tr $A(t) < 0$, we know that the volume of the image of any d-dimensional set in $\mathbb{R}^d$ obtained through a fundamental matrix solution of (2.1) is decreasing in time. We need to notice that this does not say that the solutions of (2.1) approach zero. For example, consider the system $\dot{x}_1 = -2x_1,\ \dot{x}_2 = x_2$. In this case, we have Tr $A = -1$ and so the area of a set is decreasing in time. The rectangle $\{\, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \,\}$ is mapped to the rectangle $\{\, 0 \leq x_1 \leq e^{-2t}, 0 \leq x_2 \leq e^{t} \,\}$, which becomes very long and thin.

Exercise 2.2. For the equation $\dot{x}_1 = -2x_1,\ \dot{x}_2 = x_2$, and for various types of sets which do not contain the origin, draw the images under the fundamental matrix e^{At}.

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