Fall 07 Name: ______________________________

Math 4581 Final Bonetto

1a		2c	
1b		2d	
1c		2e	
1d		2f	
1e		2g	
1f		**2h**	
2a		3a	
2b		3b	

1) The equation governing the temperature in a rod of length 1, in which an electric current is flowing, is:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + r, & 0 \le x \le 1 \\ \dfrac{\partial u}{\partial x}(0,t) = \Phi \\ \dfrac{\partial u}{\partial x}(1,t) = 0 \\ u(x,0) = u_0(x) - \dfrac{r}{2}x^2 \end{cases}$$

where

$$u_0(x) = \begin{cases} x+1 & 0 < x < \dfrac{1}{2} \\ x+2 & \dfrac{1}{2} < x < 1 \end{cases}$$

a) Under which condition on Φ does the equation admit a steady state solution? What is the physical meaning of this condition?

b) Find the steady state $v(x)$ under the condition of point (a) and write the equation for the difference $w(x,t) = u(x,t) - v(x)$. (**Hint:** you need to use conservation of energy to find the steady temperature.)

c) Write the general solution of the equation for w.

d) Find the solution for $u(x, t)$ with the given initial conditions.

e) What can you say on the convergence of the series defining $u(x, t)$ when $t = 0$? when $t = 1$?

f) (**Bonus**) Write a Fourier series for $\dfrac{\partial u}{\partial x}(x, t)$. Can you sum this series when $t = 0$? Does this series converge uniformly when $t = 1$? Justify your answer.

2) A rectangular membrane of sides 1x2 vibrates freely but the boundary is held fixed to a shaped support. The equation is thus

$$\begin{cases} \dfrac{\partial^2 u(x,y,t)}{\partial t^2} = \dfrac{\partial^2 u(x,y,t)}{\partial x^2} + \dfrac{\partial^2 u(x,y,t)}{\partial y^2} & 0 \leq x \leq 1, \quad -1 \leq y \leq 1 \\ u(x,-1,t) = -\sin(\pi x) \\ u(x,1,t) = \sin(\pi x) \\ u(0,y,t) = u(1,y,t) = 0 \\ u(x,y,0) = y\sin(\pi x) \\ \dfrac{\partial u(x,y,0)}{\partial t} = 0 \end{cases}$$

where $0 < x < 1$ and $-1 < y < 1$.

a) Write the equation for the equilibrium position $v(x,y)$ of the membrane. (**Hint:** the equilibrium position is described by a potential equation.)

b) Using separation of variable find the equilibrium position for the membrane.

c) Write the equation governing the motion of $w(x, y, t) = u(x, y, t) - v(x, y)$.

d) Use separation of variables to write the equation as an equation on t and one on x, y (eigenvalues equation). Write the general solution of the equation on t.

e) Use separation of variables again to write the eigenvalues equation as two Sturm-Liouville problems, one for x and one for y. Find the solution of the two Sturm-Liouville problems.

f) Write the general solution of the equation for $u(x, y, t)$ with an expression for the coefficients appearing in the solution.

g) Which of the coefficients of the solution $u(x, y, t)$, with the given initial condition, is non zero?

h) (**Bonus**) Identify the two components of the solution with the lowest frequencies and compute their amplitudes, *i.e.* the coefficients multiplying them in the solution.

3) A semi-infinite string in the region $0 \le x < \infty$ is held fixed at $x = 0$. The equation describing its motion is:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = 3\,\dfrac{\partial^2 u(x,t)}{\partial x^2} & x \ge 0 \\ u(0,t) = 0 & \\ u(x,0) = u_0(x) & \\ \dfrac{\partial u(x,0)}{\partial t} = 0 & \end{cases}$$

where

$$u_0(x) = \begin{cases} 0 & 0 < x < 2 \\ x - 2 & 2 < x < 3 \\ -x + 4 & 3 < x < 4 \\ 0 & x > 4 \end{cases}$$

You observe the string on the point $x = 20$.

a) Draw the solution at time $t = 1$, both position and velocity.

b) At what time will the point you observe ($x = 20$) start moving? Draw the position of the string and its velocity in that instant.