No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	55	0	15	30	100
Score:					

Question:	1	2	3	4	Total
Bonus Points:	0	10	10	0	20
Score:					

Question 1 .... *55 point*

Let $f(x)$ be the periodic function of period π given by:

$$f(x) = x\cos x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

and extended periodically to all $\mathbb{R}$.

(a) (10 points) Compute $f'(x)$ and $f''(x)$.

Solution:
Clearly we have

$$f'(x) = \cos(x) - x\sin(x) \qquad f''(x) = -2\sin(x) - x\cos(x)$$

(b) (15 points) Are f, f' and f'', piecewise continuous? continuous? piecewise smooth? (Justify your answer.)

Solution: Since

$$f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$$

we have that f is continuous. Observe that

$$f'\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

so that also f' is continuos. Moreover we have

$$f''\left(-\frac{\pi}{2}\right) = -f''\left(\frac{\pi}{2}\right) = -2$$

so that f'' is only sectionally continuous. Finally f''' exists and is continuous everywhere but for $\pi/2$ so that f'' is sectionally smooth.

(c) (15 points) Compute the Fourier series for f, f' and f'' and discuss their convergence. (Remember that

$$\sin a \cos b = (\sin(a+b) + \sin(a-b))/2$$

and

$$\int x \sin(ax)dx = -\frac{x\cos(ax)}{a} + \frac{\sin(ax)}{a^2} + C.)$$

Solution:
Clearly $f(x) = -f(-x)$ so that the F.S. contains only the sine terms. We have

$$\begin{aligned} b_n =& \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\cos(x)\sin(2nx)dx = \\ =& \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\sin((2n+1)x)dx + \int_0^{\frac{\pi}{2}} x\sin((2n-1)x)dx\right) = \\ =& \frac{2}{\pi}\left(\frac{(-1)^n}{(2n+1)^2} + \frac{-(-1)^n}{(2n-1)^2}\right) = \frac{16(-1)^{n-1}n}{\pi(4n^2-1)^2} \end{aligned}$$

Since $b_n = O(n^{-3})$ we have that the F.S. for f converges uniformly and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(2nx)$$

Thus we get that

$$f'(x) = \sum_{n=1}^{\infty} 2nb_n \cos(2nx)$$

with $nb_n = O(n^{-2})$ so that also the F.S. for f' converges uniformly. Finally

$$f''(x) = -\sum_{n=1}^{\infty} 4n^2 b_n \sin(2nx)$$

with $n^2 b_n = O(n^{-1})$. We can conclude that the F.S. for f'' converges pointwise since f'' is sectionally smooth.

(d) (15 points) Let $g(x)$ be the periodic function of period π given by:

$$g(x) = \sin x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

and extended periodically to all $\mathbb{R}$. Use the results of point (c) to find the Fourier series of g without doing integrals. (**Hint**: write g as a linear combination of f, f', and f''.)

Solution: Observe that

$$g(x) = -\frac{f''(x) + f(x)}{2}$$

so that

$$g(x) = \frac{1}{2}\sum_{n=1}^{\infty}(4n^2 - 1)b_n \sin(nx) = \sum_{n=1}^{\infty} \frac{8(-1)^{n-1}n}{\pi(4n^2 - 1)} \sin(nx)$$

2. (10 points (bonus)) Consider the heat equation for a rod of length l and heat conductivity κ:

$$\begin{cases} \frac{d}{dt}u(x,t) = \kappa \frac{d^2}{dx^2}u(x,t) \\ u(0,t) = T_0 \quad u(l,t) = T_1 \\ u(x,0) = u_0(x) \end{cases}$$

If $u(x,t)$ is a solution of the above equation, set

$$x = ly \qquad\qquad t = \frac{l^2}{\kappa}s$$

and

$$v(y,s) = u\left(ly, \frac{l^2}{\kappa}s\right).$$

Write an equation for $v(y,s)$, including boundary condition and initial condition. (**Hint**: compute $dv(y,s)/ds$ and $d^2v(y,s)/dy^2$ in term of $du(x,t)/dt$ and $d^2u(x,t)/dx^2$ and use the heat equation.)

Solution: We have

$$\begin{aligned} \frac{d}{ds}v(y,s) &= \frac{d}{ds}u\left(ly, \frac{l^2}{\kappa}s\right) = \frac{l^2}{\kappa}\dot{u}(x,t) \\ \frac{d^2}{dy^2}v(y,s) &= \frac{d^2}{dy^2}u\left(ly, \frac{l^2}{\kappa}s\right) = l^2 u''(x,t) \end{aligned}$$

Moreover

$$\begin{aligned} v(0,s) = u\left(0, \frac{l^2}{\kappa}s\right) = T_0 \\ v(1,s) = u\left(l, \frac{l^2}{\kappa}s\right) = T_1 \\ v(y,0) = u\left(ly, 0\right) = u_0(ly) \end{aligned}$$

so that v satisfies

$$\begin{cases} \frac{d}{ds}v(y,s) = \frac{d^2}{dy^2}v(y,s) \\ v(0,s) = T_0 \quad v(1,s) = T_1 \\ v(y,0) = u_0(ly) \end{cases}$$

Question 3 . . . . *15 point*

Let $f(x)$ be the function defined by the Fourier series:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$$

Answer the following questions.

(a) (15 points) Does the Fourier series for f converge uniformly? Is f continuous? (**Hint**: how big is $\sum_{n=1}^{\infty} \frac{1}{n^2}$?)

> **Solution:** Yes to both. Indeed we have that
>
> $$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$
>
> and thus, due to Theorem 1, the series converge uniformly to a continuous function.

(b) (10 points (bonus)) Is $f(x)$ sectionally smooth? That is, is $f'(x)$ sectionally continuous? (**Hint**: try to compute $f'(0)$.)

> **Solution:** Observe that $f'(x)$, if it exists, must be given by
>
> $$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)$$
>
> so that
>
> $$f'(0) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$
>
> This implies that, if $f'(x)$ exists, it cannot be sectionally continuous so that $f(x)$ is not sectionally smooth.

Question 4 . . . . *30 point*

The extremities of a rod of length a are kept at constant temperatures T_0 and T_1 while along its length it is in convective contact with a medium at a temperature that varies linearly between T_0 and T_1. This means that the temperature of the rod is governed by the equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - h\left(u(x,t) - T(x)\right) \quad 0 \le x \le a \\ u(0,t) = T_0 \\ u(a,t) = T_1 \\ u(x,0) = \frac{T_1+T_0}{2} \end{cases} \tag{1}$$

where

$$T(x) = T_0 + \frac{T_1 - T_0}{a}x.$$

and $h > 0$.

(a) (15 points) Write and solve the equation for the steady state $\bar{u}(x)$ of the rod. (**Hint:** observe that $T(x)$ is a linear function that satisfies the boundary conditions.)

Solution: The equation for the steady state is:

$$\begin{cases} \frac{\partial^2 \bar{u}(x)}{\partial x^2} = h\left(\bar{u}(x) - T(x)\right) \quad 0 \le x \le a \\ \bar{u}(0) = T_0 \\ \bar{u}(a) = T_1 \end{cases} \tag{2}$$

It is easy to find a particular solution for the non-homogenous equation:

$$\bar{u}(x) = T_0 + \frac{T_1 - T_0}{a}x.$$

Since this solution satisfies the b.c. it is the steady state.

(b) (15 points) Write the equation for the deviation $w(x,t) = u(x,t) - \bar{u}(x)$.

Solution:
The equation for the diviations is:

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = \frac{\partial^2 w(x,t)}{\partial x^2} - hw(x,t) \quad 0 \leq x \leq a \\ w(0,t) = 0 \\ w(a,t) = 0 \\ w(x,0) = \frac{T_1 - T_0}{2} - \frac{T_1 - T_0}{a} x \end{cases} \tag{3}$$