1) The equation governing the temperature $u(x,t)$ inside a rod is:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} & 0 \le x \le 1 \\ \dfrac{\partial u(0,t)}{\partial x} = ru(0,t) \\ \dfrac{\partial u(1,t)}{\partial x} = r\left(T - u(1,t)\right) \\ u(x,0) = x \end{cases}$$

a) write and solve the equation for the steady state $v(x)$.

The equation for the steady state is:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 \\ \dfrac{\partial v(0)}{\partial x} = rv(0) \\ \dfrac{\partial v(1)}{\partial x} = r\left(T - v(1)\right) \end{cases}$$

The general solution is still $v(x) = ax+b$. The first b.c. tells me that $a = rb$ while the second tells me that $a = r(T-a-b)$ or, using the other, $a = r(T-a-a/r)$ from which we get

$$a = \frac{rT}{2+r} \qquad\qquad b = \frac{T}{2+r}$$

b) write the equation for the difference $w(x,t) = u(x,t) - v(x)$.

The equation for w is the homogeneous version of that for u so that:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \dfrac{\partial^2 w(x,t)}{\partial x^2} & 0 \le x \le 1 \\ \dfrac{\partial w(0,t)}{\partial x} = rw(0,t) \\ \dfrac{\partial w(1,t)}{\partial x} = -rw(1,t) \\ u(x,0) = \left(1 - \dfrac{rT}{2+r}\right) x - \dfrac{T}{2+r} \end{cases}$$

c) use separation of variable to reduce the problem to a Sturm-Luiville problem. Find the eigenvalues and eigenfunctions. Explain why you can expand in eigenfunctions. Write the general solution for $w(x,t)$ and an expression for the coefficient in term of $w(x,0)$.

Writing $w(x,t) = T(t)s(x)$ we get the equation

$$\begin{cases} \dfrac{\partial T(t)}{\partial t} = \mu T(t) \\ \dfrac{\partial^2 s(x)}{\partial x^2} = \mu s(x) \\ s'(0) - rs(0) = 0 \\ s'(1) + rs(1) = 0 \end{cases}$$

Observe that the Theorem on section 2.8 tells you that all μ are non negative so that I can write $\mu = -\lambda^2$. The general solution of the equation for $s(x)$ is $s(x) = a\cos(\lambda x) + b\sin(\lambda x)$ so that $s'(x) = -a\lambda\sin(\lambda x) + b\lambda\cos(\lambda x)$. The first b.c. tells me $ra = \lambda b$ and the second tells me

$$\lambda b\cos(\lambda) + rb\sin(\lambda) = \frac{\lambda^2}{r}b\sin(\lambda) - b\lambda\cos(\lambda x)$$

that gives

$$\tan(\lambda) = \frac{2r\lambda}{\lambda^2 - r^2}$$

Observe that

$$\lim_{\lambda\to\infty}\frac{2r\lambda}{\lambda^2 - r^2} = 0$$

so that we have infinitely many solution λ_n and $\lim_{n\to\infty}\lambda_n = n\pi$. Finally we get

$$s_n(x) = \lambda_n\cos(\lambda_n x) + r\sin(\lambda_n x)$$

From the general theory we know that the $s_n(x)$ are orthogonal because they are the eigenvalue of a regular Sturm-Luoiville problem. Setting:

$$c_n = \int_0^1 s_n^2(x)dx$$

We have, for every function $f(x)$, that

$$f(x) = \sum a_n s_n(x)$$

where

$$a_n = \int_0^1 f(x)s_n(x)dx.$$

So we obtain that the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} s_n(x)$$

and setting

$$a_n = \frac{1}{c_n} \int_0^1 \left[\left(1 - \frac{rT}{2+r} \right) x - \frac{T}{2+r} \right] s_n(x) dx$$

we obtain a solution for our problem.

e) Give an estimate from above and below of the first eigenvalue. How long do you have to wait to be sure that $|w(x,t)| \leq 10^{-3}$. Use only the series truncated at the first term but observe that you need an estimate of the first coefficient.

Observe that the function

$$g(\lambda) = \frac{2r\lambda}{\lambda^2 - r^2}$$

is negative for for $\lambda \leq r$ and positive after. Moreover $\lim\limits_{\lambda \to r-} = -\infty$ and $\lim\limits_{\lambda \to r+} = +\infty$. Finally $g(0) = 0$. This implies that if $0 < r < \pi/2$ than $r < \lambda_1 < \pi/2$, otherwise $\pi/2 < \lambda_1 < \pi$. Writing the truncated solution we have

$$w(x,t) \simeq a_1 e^{-\lambda_1^2 t} s_1(x)$$

Observe that $|s_1(x)| \leq \lambda_1 + r$ so that we have to find t such that

$$|a_1| e^{-\lambda_1^2 t}(\lambda_1 + r) \leq 10^{-3}$$

that is

$$t > \frac{\ln\left(1000(r + \lambda_1)|a_1|\right)}{\lambda_1}$$

f) **Bonus**: write the solution of the problem. Remember that

$$\int x\cos(\lambda x)dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}$$
$$\int x\sin(\lambda x)dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}$$

We have to compute

$$\int_0^1 s_n(x)dx = \int_0^1 \left(\lambda_n \cos(\lambda_n x) + r\sin(\lambda_n x)\right) dx = \sin(\lambda_n) - r\frac{\cos(\lambda_n) - 1}{\lambda_n} = d_n$$

and

$$\int_0^1 x s_n(x)dx = \int_0^1 \left(\lambda_n x\cos(\lambda_n x) + rx\sin(\lambda_n x)\right) dx =$$
$$= \left(\frac{\cos(\lambda_n x)}{\lambda_n} + x\sin(\lambda_n x) + \frac{r\sin(\lambda_n x)}{\lambda_n^2} - \frac{rx\cos(\lambda_n x)}{\lambda_n}\right)\Bigg|_0^1 =$$
$$= \frac{1-r}{\lambda_n}\cos\lambda_n + \left(1 + \frac{r}{\lambda_n^2}\right)\sin\lambda_n - \frac{1}{\lambda_n} = e_n$$

Finally we have

$$c_n = \int_0^1 \left(\frac{\lambda_n^2 - r^2}{2}\cos(2\lambda_n x) + \frac{\lambda_n^2 + r^2}{2} + r\lambda_n\sin(2\lambda_n x)\right) dx =$$
$$= \frac{r^2\lambda_n^2 - 1}{2\lambda_n}\sin(2\lambda_n) - r\left(\cos(2\lambda_n) - 1\right) + \frac{r^2\lambda_n^2 + 1}{2}$$

so that

$$a_n = \left(1 - \frac{rT}{2+r}\right)\frac{e_n}{c_n} - \frac{T}{2+r}\frac{d_n}{c_n}$$

2) Let $f(x)$ a continuous and differentiable function defined for all x. Assume that

$$|f(x)| \leq Ce^{-\lambda|x|}$$

with C and λ positive. Finally let

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} f(x)dx. \tag{1}$$

Consider now the function

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+nL)$$

with $L > 0$.

a) Show that $F(x)$ exists and it is periodic of period L.

Observe that

$$\begin{aligned} F(x+L) &= \sum_{n=-\infty}^{\infty} f(x+L+nL) = \\ &= \sum_{n=-\infty}^{\infty} f(x+(n+1)L) = \sum_{m=-\infty}^{\infty} f(x+mL) = F(x) \end{aligned}$$

so that $F(x)$ is periodic of period L. Let now $0 < x < L$. We have

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+nL) \leq C \sum_{n=-\infty}^{\infty} e^{-\lambda|x+nL|} \leq Ce^{\lambda x} \sum_{n=-\infty}^{\infty} e^{-\lambda|n|L} < +\infty$$

where we used that $|x+nL| \geq |nL| - |x|$ so that

$$e^{-\lambda|x+nL|} \leq e^{\lambda x} e^{-\lambda|nL|}.$$

b) Let

$$F(x) = \sum c_n e^{i\frac{2n\pi}{L}x}.$$

Find the coefficients c_n. (**Hint**: write an expression for c_n as a sum of integrals and than change variable $y = x + nL$ and ...)

$$\begin{aligned}
c_m =& \frac{1}{L}\int_0^L e^{-i\frac{2n\pi}{L}x}F(x)dx = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_0^L e^{-i\frac{2n\pi}{L}x}f(x+nL)dx = \\
=& \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_{nL}^{(n+1)L} e^{-i\frac{2n\pi}{L}(y-nL)}f(y)dy = \frac{1}{L}\int_{-\infty}^{\infty} e^{-i\frac{2n\pi}{L}y}f(y)dy = \\
=& \hat{f}\left(-\frac{2n\pi}{L}\right)
\end{aligned}$$