5. Solve the problem

$$\begin{array}{llll} \nabla^2 u = 0, & & 0 < x < a, & 0 < y < b, \\ u(0, y) = 0, & u(a, y) = 0, & 0 < y < b, & \\ u(x, 0) = 0, & u(x, b) = f(x), & 0 < x < a, & \end{array}$$

where f is the same as in the example. Sketch some level curves of $u(x, y)$.

6. Solve the potential problem on the rectangle $0 < x < a$, $0 < y < b$, subject to the boundary conditions $u(a, y) = 1$, $0 < y < b$, and $u = 0$ on the rest of the boundary.

7. Solve the problem of the potential equation in the rectangle $0 < x < a$, $0 < y < b$, for each of the following sets of boundary conditions. Before solving, make a pictorial version of the problem as in Exercise 9 of Section 4.1.

   a. $u(x, b) = 100$, $0 < x < a$; $u = 0$ on the other three sides of the rectangle.

   b. $u(x, b) = 100$, $0 < x < a$; $u(a, y) = 100$, $0 < y < b$; $u = 0$ on the other two sides of the rectangle.

   c. $u(x, b) = bx$, $0 < x < a$; $u(a, y) = ay$, $0 < y < b$; $u = 0$ on the other two sides of the rectangle.

8. Solve the problem for u_2. (That is, derive Eq. (18).)

4.3 Further Examples for a Rectangle

In Section 4.2, we solved Dirichlet problems with separation of variables. The same method applies to problems with other types of boundary conditions, as shown in the following.

Example 1.
In this problem, the unknown function might be a voltage in a conductor. The left and right sides are electrically insulated.

$$\begin{array}{llll} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & 0 < x < a, & & 0 < y < b, \\ \dfrac{\partial u}{\partial x}(0, y) = 0, & & \dfrac{\partial u}{\partial x}(a, y) = 0, & 0 < y < b, \\ u(x, 0) = 0, & & u(x, b) = V_0 x/a, & 0 < x < a. \end{array}$$

We have homogeneous conditions on the facing sides at $x = 0$ and $x = a$. If we look for solutions in the product form $u(x, y) = X(x)Y(y)$, we find (as expected) that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{constant}.$$

The conditions at $x = 0$ and $x = a$ become

$$X'(0) = 0, \quad X'(a) = 0.$$

If we make the separation constant $-\lambda^2$, we find a familiar eigenvalue problem for X whose solution is

$$X_0(x) = 1, \quad \lambda_0 = 0,$$

$$X_n(x) = \cos(\lambda_n x), \quad \lambda_n = n\pi/a, \quad n = 1, 2, \ldots.$$

For the factor $Y(y)$, the differential equation is

$$Y_0'' = 0, \quad \text{or} \quad Y_n'' - \lambda_n^2 Y_n = 0$$

with solution

$$Y_0(y) = a_0 + b_0 y \quad \text{or} \quad Y_n(y) = a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y).$$

Thus, the principle of superposition leads to the series solution

$$u(x, y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \big(a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)\big) \cos(\lambda_n x).$$

The boundary condition at $y = 0$ becomes

$$a_0 + \sum_{1}^{\infty} a_n \cos(\lambda_n x) = 0, \quad 0 < x < a,$$

from which we see that all the a's are 0. Then at $y = b$ we have

$$b_0 b + \sum_{n=1}^{\infty} \big(b_n \sinh(\lambda_n b)\big) \cos(\lambda_n x) = \frac{V_0 x}{a}, \quad 0 < x < a.$$

This is a slightly disguised cosine series. The coefficients are

$$b_0 b = \frac{1}{a} \int_0^a V_0 \left(\frac{x}{a}\right) dx,$$

$$b_n \sinh(\lambda_n b) = \frac{2}{a} \int_0^a V_0 \left(\frac{x}{a}\right) \cos(\lambda_n x)\, dx.$$

See a color graphic of the solution on the CD. □

We have seen that the success of the separation of variables method depends on having homogeneous boundary conditions at the ends of one of the intervals involved. In Section 4.2 we mentioned splitting up a Dirichlet problem, if necessary, to achieve this. The same splitting technique applies in problems where boundary condition of other kinds are used. The principle is to zero conditions on two facing sides of the region and to copy the rest.

Example 2.

This problem may describe the temperature $u(x, y)$ in a thin plate between insulating sheets.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \qquad 0 < y < b,$$
$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad u(a, y) = Sy, \qquad 0 < y < b,$$
$$\frac{\partial u}{\partial y}(x, 0) = S, \qquad u(x, b) = \frac{Sbx}{a}, \quad 0 < x < a.$$

Since we have nonhomogeneous conditions on adjacent sides, we must split the problem in order to solve by separation of variables. Here are the two problems:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \qquad\qquad \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0,$$
$$\frac{\partial u_1}{\partial x}(0, y) = 0, \quad u_1(a, y) = 0, \qquad \frac{\partial u_2}{\partial x}(0, y) = 0, \quad u_2(a, y) = Sy,$$
$$\frac{\partial u_1}{\partial y}(x, 0) = S, \quad u_1(x, b) = \frac{Sbx}{a}, \qquad \frac{\partial u_2}{\partial y}(x, 0) = 0, \quad u_2(x, b) = 0.$$

The solution of the original problem is the sum $u = u_1 + u_2$. Here is the reasoning in detail.

1. The potential equation is linear and homogeneous. By the Principle of Superposition, the sum of solutions is a solution.
2. At $x = a$ we have $u(a, y) = 0 + Sy$, and at $y = b$ we have $u(x, b) = Sbx/a + 0$. Both conditions are satisfied.
3. From elementary calculus, we know

$$\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y}.$$

Then at the left and bottom boundaries, we have

$$\frac{\partial u}{\partial x}(0, y) = 0 + 0, \quad \frac{\partial u}{\partial y}(x, 0) = S + 0.$$

These are satisfied as well.

Thus, it remains to solve the two problems for u_1 and u_2. (See the Exercises.) Here are product solutions. For u_1:

$$\cos(\lambda_n x)\big(a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)\big), \quad \lambda_n = \left(n - \frac{1}{2}\right)\frac{\pi}{a}, \quad n = 1, 2, \ldots.$$

For u_2:

$$\cos(\mu_n y)\big(A_n \cosh(\mu_n x) + B_n \sinh(\mu_n x)\big), \quad \mu_n = \left(n - \frac{1}{2}\right)\frac{\pi}{b}, \quad n = 1, 2, \ldots.$$

□

The simple polynomial solutions that we found in Section 4.1, Exercise 1, can be very useful in reducing the number of series needed for a solution. If nonhomogeneous conditions are given on adjacent sides and these are constants or first-degree polynomials in one variable, then a polynomial may be able to satisfy enough of them to simplify the work.

Example 3.
Refer to the problem in Example 2. The polynomial $v(y) = Sy$ satisfies the potential equation and several of the boundary conditions:

$$\frac{\partial v}{\partial x}(0, y) = 0, \quad v(a, y) = Sy, \quad 0 < y < b,$$

$$\frac{\partial v}{\partial y}(x, 0) = S, \quad v(x, b) = Sb, \quad 0 < x < a.$$

Thus, we may set $u(x, y) = v(y) + w(x, y)$ and determine that w must be the solution of this problem, similar to the problem for u_2 in Example 2:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$$

$$\frac{\partial w}{\partial x}(0, y) = 0, \quad w(a, y) = 0, \quad 0 < y < b,$$

$$\frac{\partial w}{\partial y}(x, 0) = 0, \quad w(x, b) = \frac{Sb(x - a)}{a}, \quad 0 < x < a.$$

The solution is left as an exercise. □

Poisson Equation

Many problems in engineering and physics require the solution of the Poisson equation,

$$\nabla^2 u = -H \quad \text{in a region } \mathcal{R}.$$

Here are three examples of such problems.

(1) u is the deflection of a membrane that is fastened at its edges, so $u = 0$ on the boundary of $\mathcal{R}$; H is proportional to the pressure difference across the membrane. (See Section 5.1.)

(2) u is the steady-state temperature in a cross section of a long cylindrical rod that is carrying an electrical current; H is proportional to the power in resistance heating. (See Section 5.2.)

(3) u is the stress function on the cross section $\mathcal{R}$ of a cylindrical bar or rod in torsion (the shear stresses are proportional to the partial derivatives of u); H is proportional to the rate of twist and to the shear modulus of the material; $u = 0$ on the boundary of $\mathcal{R}$.

If H is a constant, a polynomial of the form

$$P(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$$

is a solution of Poisson's equation, provided that

$$2(D + F) = -H.$$

The other coefficients are arbitrary and may be chosen for convenience in satisfying boundary conditions.

Example 4.
Find the deflection u of a membrane that is modeled by this problem. The constant is $H = p/\sigma$, where p is the pressure difference (below to above) and σ is the surface tension in the membrane.

A polynomial can be chosen that satisfies the partial differential equation and boundary conditions on facing sides. For instance,

$$v(x) = \frac{Hx(a - x)}{2}$$

satisfies the Poisson equation and two boundary conditions,

$$v(0) = 0, \quad v(a) = 0.$$

physics require the solution of the Poisson

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$= \frac{Hx(a-x)}{2}$

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$= 0, \quad v(a) = 0.$

Thus, we may set $u(x, y) = v(x) + w(x, y)$ and determine that w is a solution of the problem

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$$

$$w(0, y) = 0, \quad w(a, y) = 0, \quad 0 < y < b,$$

$$w(x, 0) = -v(x), \quad w(x, b) = -v(x), \quad 0 < x < a.$$

The CD has color graphics of the solution. □

In general, if H is a polynomial in x and y, a solution can be found in the form of a polynomial of total degree 2 higher than H. If H is a more general function, it may be expressed as a double Fourier series (see Chapter 5), and the partial differential equation can be solved following the idea of Section 1.11B.

EXERCISES

1. Solve the problem consisting of the potential equation on the rectangle $0 < x < a$, $0 < y < b$ with the given boundary conditions. Two of the three are very easy if a polynomial is subtracted from u.

   a. $\frac{\partial u}{\partial x}(0, y) = 0$; $\quad u = 1$ on the remainder of the boundary.

   b. $\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(a, y) = 0; \quad u(x, 0) = 0, \quad u(x, b) = 1.$

   c. $\frac{\partial u}{\partial x}(x, 0) = 0, \quad u(x, b) = 0; \quad u(0, y) = 1, \quad u(a, y) = 0.$

2. Same task as Exercise 1.

   a. $u(x, b) = 100$; the outward normal derivative is 0 on the rest of the boundary.

   b. $u(x, b) = 100, \quad u(0, y) = 0, \quad u(a, y) = 100, \quad \frac{\partial u}{\partial y}(x, 0) = 0.$

3. Finish the work for Example 1: Find the b_n, form the series, and check that all conditions are satisfied.

4. In Example 2, check that the given product solution for $u_1(x, y)$ satisfies the conditions and determine the coefficients a_n and b_n.

5. In Example 2, check that the given product solution for $u_2(x, y)$ satisfies the conditions and determine the coefficients A_n and B_n.

6. Explain the difference between the cosine series in Example 1 and the cosine series for $u_1(x, y)$ in Example 2. What is the source of the difference?

7. Finish the work for Example 3. That is, find $w(x, y)$ as a series and check that the boundary conditions are all satisfied.

8. Compare the amount of work involved in solving the problem of Example 2 (including Exercises 4 and 5) with the work for Example 3 (including Exercise 7).

9. Finish the work of Example 4: Find the solution, as a series, for $w(x, y)$. Form $u(x, y)$ and use the first term of the series for $w(x, y)$ to obtain an expression for the value of $u(\frac{a}{2}, \frac{b}{2})$. This would be the maximum deflection of the membrane.

10. Find the condition on the coefficients so that the following general second-degree polynomial is a solution of the Poisson equation, $\nabla^2 p = -H$, where H is constant:

$$p(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2.$$

11. Same task as Exercise 10, but $H = K(x^2 + y^2)$ and p is this part of the general fourth-degree polynomial

$$p(x, y) = Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4,$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = H, \quad 0 < x < a, \quad 0 < y < b,$$

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < x < a,$$

$$u(x, 0) = 0, \quad u(x, b) = 0, \quad 0 < y < b.$$

4.4 Potential in Unbounded Regions

The potential equation, as well as the heat and wave equations, can be solved in unbounded regions. Consider the following problem, in which the region involved is half a vertical strip, or a slot:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y, \tag{1}$$

$$u(x, 0) = f(x), \quad 0 < x < a, \tag{2}$$

$$u(0, y) = g_1(y), \quad 0 < y, \tag{3}$$

$$u(a, y) = g_2(y), \quad 0 < y. \tag{4}$$

As usual, we required that $u(x, y)$ remain bounded as $y \to \infty$.