Fall 04 | Name: ____________________
Math 4581 | Test 1 | Bonetto

1) A rod of circumference 1 and length a looses heat at a constant rate per unit surface r through its cylindrical surface. Moreover one extremity of the rod is insulated while the heat flowing in the rod at the other is fixed to $\Phi > 0$. Thus, the equation governing the temperature $u(x,t)$ inside the rod is:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} - r \qquad 0 \le x \le a \\ \dfrac{\partial u(0,t)}{\partial x} = 0 \\ \dfrac{\partial u(a,t)}{\partial x} = \Phi \\ u(x,0) = T \end{cases}$$

a) write the equation for the steady state $v(x)$. Under which condition on r can it be solved? Can you give a physical meaning to this condition? (**Hint**: what does ra represent?)

b) solve the steady state equation.(**Hint**: how does the total heat in the system change in time?)

b) write the equation for the difference $w(x, t) = u(x, t) - v(x)$.

c) use separation to find the general solution for $w(x, t)$.

e) write the solution of the problem.(**Hint**: use that:

$$\int x^2 \cos(\lambda x) dx = \frac{2x\cos(\lambda x)}{\lambda^2} - \frac{(2-\lambda^2 x^2)\sin(\lambda x)}{\lambda^3}$$

or use derivatives and Dirac δ function, but pay attention.)

2) An infinite rod is initially at temperature $u(x,0) = \exp(-x^2)$. Thus, the equation governing the temperature $u(x,t)$ inside the rod is:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t} & -\infty \le x \le \infty \\ u(x,0) = e^{-x^2} & \end{cases}$$

Write the solution $u(x,t)$ for any t. (**Hint**. observe that the initial condition is the solution at some time t_0 of the equation:

$$\begin{cases} \dfrac{\partial^2 v(x,t)}{\partial x^2} = \dfrac{\partial v(x,t)}{\partial t} & -\infty \le x \le \infty \\ v(x,0) = a\delta(x) & \end{cases}$$

Find a and t_0 such that $u(x,0) = v(x,t_0)$ and ...)