n. 1 sec 5.2

The equation is

$$v''(x) - \gamma^2 v(x) = \gamma^2 U$$
$$u(0,t) = T_0 \qquad u(a,t) = T_1$$

The Homogeneous equation is:

$$v''(x) - \gamma^2 v(x) = 0$$

whose general solution is:

$$v(x) = b_1 \cosh(\gamma x) + b_2 \sinh(\gamma x)$$

while a particular solution is

$$v(x) = U$$

Thus

$$v(x) = b_1 \cos(\gamma x) + b_2 \sin(\gamma x) + U$$

The B.C. give:

$$T_0 = v(0) = b_1 + U \Rightarrow b_1 = U - T_0$$
$$T_1 = v(a) = U + (U - T_0)\cosh(\gamma a) + b_2 \sinh(\gamma a) \Rightarrow b_2 = \frac{U + (U - T_0)\cosh(\gamma a)}{\sinh(\gamma a)}$$

n. 3 sec 5.2

All as before for the general solution:

$$v(x) = T + b_1 \cosh(\gamma x) + b_2 \sinh(\gamma x)$$

Then we get:

$$T = v(0) = b_1 + T \Rightarrow b_1 = 0$$
$$T = v(a) = T + b_2 \sinh(\gamma a) \Rightarrow b_2 = 0$$

From the solution I guess that there is a typo and the author meant the equation:

$$\partial_x u(x,t) + \gamma^2 (u(x,t) - T) = \frac{1}{k} \partial_t u(x,t)$$

In this case you get a general solution:

$$v(x) = T + b_1 \cos(\gamma x) + b_2 \sin(\gamma x)$$

and, if $\sin \gamma a \neq 0$, *i.e.* $\gamma = n\pi/a$, then

$$T = v(0) = b_1 + T \Rightarrow b_1 = 0$$
$$T = v(a) = T + b_2 \sin(\gamma a) \Rightarrow b_2 = 0$$

if $\sin \gamma a = 0$ there is no condition for b_2 so that the solution is not unique. Observe that in this case when you do separation of variable for the remaining eqaution you will find that $\sinh(\gamma a)$ is an solution with eigenvalue $\lambda = 0$. There is no contradiction with our statements because the condution $\alpha_1, \alpha_2 > 0$ is not satisfied.

n. 7 sec 5.2

The equation is

$$v''(x) + r = 0$$

The homogeneous eqaution is:

$$v''(x) = 0$$

with general solution $v(x) = b_1 x + b_2$ while a particular solution of the non homogeneous is $v(x) = rx^2/2$ so that the general solution is

$$v(x) = b_1 x + b_2 + \frac{r}{2} x^2$$

The B.C. give

$$T_0 = v(0) = b_2 \Rightarrow b_2 = T_0$$
$$0 = v'(a) = b_1 + ra \Rightarrow b_2 = -ra$$

n. 9 sec 5.2

Steady state equation:

$$Dv''(x) - Sv'(x) = 0$$

is already homogeneous and has a general solution:

$$v(x) = b_1 + b_2 e^{\frac{S}{D} x}$$

So that

$$U = v(0) = b_1 + b_2 \Rightarrow b_1 = -b_2 + U$$
$$0 = v(a) = b_1 + b_2 e^{\frac{S}{D} a} \Rightarrow b_1 = b_2 e^{\frac{S}{D} a}$$

Solving this simple system gives you the solution

n. 3 sec 2.3

Call

$$v(\xi,\tau) = w\left(\frac{x}{a}, \frac{kt}{a^2}\right)$$

then we have

$$\partial_\xi v(\xi,\tau) = \frac{\partial x}{\partial \xi}\partial_x v(\xi,\tau) = a\partial_x w(x,t)$$

In the same way we get

$$\partial_\xi^2 v(\xi,\tau) = a^2\partial_x w(x,t) \qquad \partial_\tau v(\xi,\tau) = \frac{a^2}{k}\partial_t w(x,t)$$

Thus it follows that:

$$\partial_\xi^2 v(\xi,\tau) = \partial_t w(x,t).$$

n. 9 sec 2.3

We know that the steady state solution is linear so that we get

$$v(x) = C_1$$

The equation for $w(x,t) = C(x,t) - v(x)$ becomes:

$$\begin{cases} \partial_x^2 w(x,t) = D\partial_t w(x,t) \\ w(0,t) = w(a,t) = 0 \\ w(x,0) = C_0 - C_1 \end{cases}$$

To solve the equation we must write:

$$C_0 - C_1 = \sum_{n=0}^{\infty} b_n sin\left(\frac{n\pi}{a}x\right)$$

we know that this gives

$$b_n = (C_0 - C_1)\frac{2(1-(-1)^n)}{\pi n}$$

Then the solution is:

$$w(x,t) = \sum_{n=0}^{\infty} b_n sin\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2 k}{a^2}t\right)$$

We have that $C(a/2,t) = C_1 + w(a/2,t)$ so that the equation is:

$$C_1 + w(a/2,t) - C_0 = 0.9(C_1 - C_0)$$

from which

$$w(a/2,t) - C_0 = -0.1(C_1 - C_0)$$

Let approximate $w(x,t)$ with:

$$w(x,t) = b_1 sin\left(\frac{\pi}{a}x\right)\exp\left(-\frac{\pi^2 k}{a^2}t\right) = \frac{4(C_0 - C_1)}{\pi}\exp\left(-\frac{\pi^2 k}{a^2}t\right)$$

so that the equation is:

$$\frac{40}{\pi} = \exp\left(\frac{\pi^2 k}{a^2}t\right) \qquad t = \frac{a^2}{\pi^2 k}\log\left(\frac{40}{\pi}\right)$$

n. 1 sec 2.4

Observe that the SS solution is:

$$v(x) = T_1/2$$

We must write:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right)$$

This is a cosine Fourier series and gives:

$$a_n = 2T_1\frac{(-1)^n - 1}{n^2\pi^2}$$

so that the solution is

$$u(x,t) = \frac{T_1}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{\pi^2 n^2 k}{a^2}t\right)$$

Observe that calling $a_0 = T_1/2$ I can write the above solution as:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{\pi^2 n^2 k}{a^2}t\right)$$

n. 5 sec 2.4

The steady state equation is:

$$v''(x) = 0$$

whose general solution is:

$$v(x) = ax + b$$

The condition at 0 implies that $a = 0$ so that $v(x) = b$. The only possibility is that $S_0 = S_1$. Indeed if $S_0 - S_1 \neq 0$ thers would be a net flow of energy in (or out) the rod. In this situation the energy content of the rod keep changing and no steady stae can be reached.

From the B.C. we $v(x) = Sx + b$, where b is fixed by:

$$\int_0^a v(x)dx = \int_0^a f(x)dx$$

Moreover

$$w'(0,t) = u'(0,t) - v'(0) = 0$$

same for a.

It is easy to show that $u(x,t) = A(kt - x^2/2) + Bx$ solve the equation. Moreover from B.C. we get $B = S_0$ and $A = (S_0 - S_1)/a$ so that $u(x,t)$ goes to $+\infty$ or $-\infty$ depending on the sign of $S_0 - S_1$.

n. 8 sec 2.4

The equation has solution:

$$\phi(x) = b_1 \cosh(px) + b_2 \sinh(px)$$

so we get

$$\begin{aligned} 0 = \phi'(0) = b_2 \cosh(0) \Rightarrow b_2 = 0 \\ 0 = \phi'(a) = b_1 \sinh(pa) \Rightarrow b_1 = 0 \end{aligned}$$

n. 3 sec 2.5

The function $G(x)$ is obtained reflecting $g(x)$ around a. I can the take the odd extension around 0 and take the Fourier Series. This will contain only sine terms. Moreover this is now a function periodic of period $4a$ so that I have

$$G(x) = \sum_{n=0}^{\infty} B_N \sin\left(\frac{N2\pi}{4a}x\right)$$

n. 4 sec 2.5

We have

$$
\begin{aligned}
&B_N = \frac{1}{a}\int_0^{2a} G(x)\sin\left(\frac{N2\pi}{4a}x\right)dx = \\
&\frac{1}{a}\int_0^{a} g(x)\sin\left(\frac{N2\pi}{4a}x\right)dx + \frac{1}{a}\int_0^{a} g(2a-x)\sin\left(\frac{N2\pi}{4a}(2a-x)\right)dx = \\
&\frac{(1-(-1)^n)}{a}\int_0^{a} g(x)\sin\left(\frac{N2\pi}{4a}x\right)dx
\end{aligned}
$$

n. 6 sec 2.5

If $\lambda_n \neq \lambda_m$ then

$$
2\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx = \int_0^a \cos((\lambda_n-\lambda_m)x)dx - \int_0^a \cos((\lambda_n+\lambda_m)x)dx = \left(\frac{\sin((\lambda_n-\lambda_m)x)}{\lambda_n-\lambda_m} - \frac{\sin((\lambda_n+\lambda_m)x)}{\lambda_n+\lambda_m}\right)\Bigg|_0^a
$$

Observe that $\lambda_n - \lambda_m = (n-m)\pi/a$ and $\lambda_n + \lambda_m = (n+m-1)\pi/a$ form which

$$
\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx = 0
$$

.

If $\lambda_n = \lambda_m$ then

$$
\int_0^a \sin^2(\lambda_n x)dx = 1/2\int_0^a (\sin^2(\lambda_n x)+\cos^2(\lambda_n x))dx = \frac{a}{2}
$$

n. 3 sec 2.6

The eigenvalue are $\mu_n = -\lambda_n^2$. The solution

$$
\tan(\lambda a) = \frac{-\kappa}{a}
$$

are the same as those of

$$
\tan(\lambda a) = \frac{\kappa}{a}
$$

but for the sign. So they do not gine new μ_n.