No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	30	20	20	10	80
Score:					

Question:	1	2	3	4	Total
Bonus Points:	0	10	10	0	20
Score:					

Question 1 . . . . *30 point*

Consider the differential equation

$$\dot{x} = -x^3 + ax \tag{1}$$

where a is a real number.

(a) (10 points) For all possible values of a, find the fixed points and determine whether they are sinks or sources. Find the value of a for which there is a bifurcation.

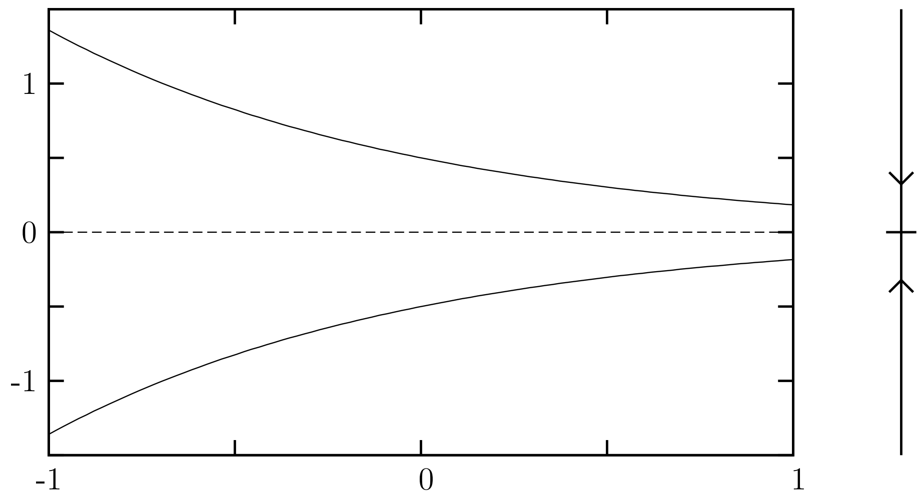
Solution: The fixed points are the solutions of $-x^3 + ax = 0$. If $a < 0$ there is only one solution $x = 0$. If $a > 0$ there are 3 solutions: $x = 0$ and $x = \pm\sqrt{a}$.

Form the derivative we see that $x = 0$ is a sink in $a < 0$ and a source if $a > 0$ while $x = \pm\sqrt{a}$ are sinks for all $a > 0$. Moreover for $a = 0$, $-x^3$ is positive for x negative and negative for x positive so that $x = 0$ is a sink for $a = 0$.

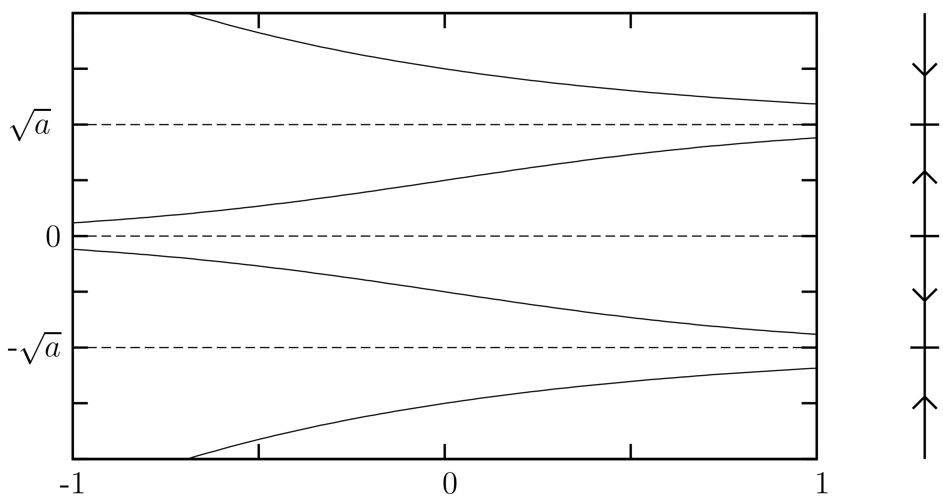
Summarizing, $a = 0$ is a bifurcatrion. For $a \leq 0$ there is only one fixed point at $x = 0$ and it is a sink. For $a > 0$ there are 3 fixed points, one source at $x = 0$ and two sinks at $x = \pm\sqrt{a}$.

(b) (10 points) Sketch the solution graphs and the phase line of (1) for a before and after the bifurcation.

Solution: For $a \leq 0$ the solution graphs and the phase line look like:

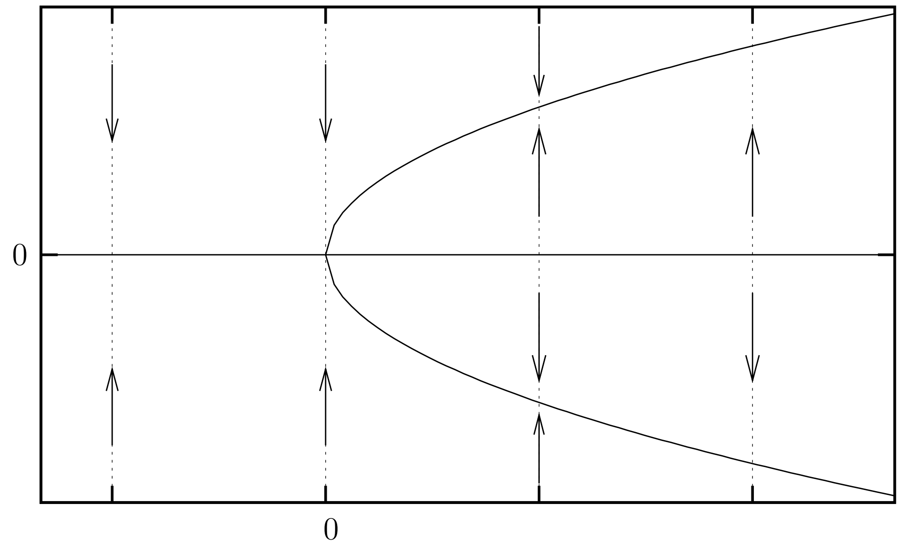


While for $a > 0$ the solution graphs and the phase line look like:



(c) (10 points) Draw a bifurcation diagram for (1).

Solution: The bifurcation diagram looks like:



Question 2 .... *20 point*

Consider the system

$$\dot{X} = AX \tag{2}$$

where

$$A = \begin{pmatrix} 1+3a & -2 \\ 4a^2 & 1-3a \end{pmatrix}$$

where a is a real number.

(a) (10 points) Find the general solution of (2) for $a \neq 0$.

Solution:

The eigenvalues are the solutions of:

$$\lambda^2 - 2\lambda + (1 - a^2) = 0$$

so that

$$\lambda_\pm = 1 \pm a.$$

The relative eigenvectors are

$$V_+ = \begin{pmatrix} 1 \\ a \end{pmatrix} \qquad V_- = \begin{pmatrix} 1 \\ 2a \end{pmatrix}$$

The general solution is

$$X(t) = c_1 e^{(1+a)t} \begin{pmatrix} 1 \\ a \end{pmatrix} + c_2 e^{(1-a)t} \begin{pmatrix} 1 \\ 2a \end{pmatrix}$$

(b) (10 points) Find the general solution of (2) for $a = 0$.

Solution: For $a = 0$ the matrix A becomes:

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Thus

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an eigenvector. The vector

$$V_2 = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

satisfies

$$AV_2 = V_2 + V_1$$

so that the general solution is

$$X(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} t \\ -\frac{1}{2} \end{pmatrix}$$

(c) (10 points (bonus)) Let $X_a(t)$ be the solution of (2) satisfying $X_a(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Show that $X_a(t)$ is continuous in a for every t. (**Hint**: remember that $\lim_{a\to 0}(e^{at} - e^{-at})/a = 2t$)

Solution: The only place where we can have problem is for $a = 0$. For $a \neq 0$ we have that

$$X_a(t) = -\frac{1}{a}e^{(1+a)t}\begin{pmatrix} 1 \\ a \end{pmatrix} + \frac{1}{a}e^{(1-a)t}\begin{pmatrix} 1 \\ 2a \end{pmatrix} = e^t\begin{pmatrix} \frac{e^{-at}-e^{at}}{a} \\ -e^{-at} + 2e^{at} \end{pmatrix}$$

so that

$$\lim_{a\to 0} X_a(t) = e^t\begin{pmatrix} -2t \\ 1 \end{pmatrix}$$

On the other hand, from point (b) we get

$$X_0(t) = -2e^t\begin{pmatrix} t \\ -\frac{1}{2} \end{pmatrix}$$

so that

$$\lim_{a\to 0} X_a(t) = X_0(t)$$

and $X_a(t)$ is continuous for every t.

Question 3 .... *20 point*

Consider the differential equation:

$$\dot{X} = AX \tag{3}$$

where

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

(a) (10 points) Let $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution of (3). Call

$$\rho(t) = \sqrt{x_1(t)^2 + x_2(t)^2}$$

Show that:

$$\dot{\rho} = -\rho$$

Solution: Differentiating we get

$$\dot{\rho} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{\sqrt{x_1^2 + x_2^2}} = \frac{x_1(-x_1 + x_2) + x_2(-x_1 - x_2)}{\sqrt{x_1^2 + x_2^2}} = -\frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2}} = -\rho$$

where we used that $\dot{x}_1 = -x_1 + x_2$ and $\dot{x}_2 = -x_1 - x_2$.

(b) (10 points) Show that the function $H(x_1, x_2) = (y_1, y_2)$ defined by

$$\begin{cases} y_1 &= \cos(\ln(\rho))x_1 + \sin(\ln(\rho))x_2 \\ y_2 &= -\sin(\ln(\rho))x_1 + \cos(\ln(\rho))x_2 \end{cases}$$

is a conjugacy between (3) and

$$\dot{Y} = BY \tag{4}$$

with

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Here, like in point (a), $\rho = \sqrt{x_1^2 + x_2^2}$. (**Hint:** Compute first $\frac{d}{dt}\ln(\rho)$ and use it to compute $\dot{y}_1$ and $\dot{y}_2$ and show that they satisfy (4).)

Solution: First we have

$$\frac{d}{dt}\ln(\rho) = \frac{\dot{\rho}}{\rho} = -1$$

so that

$$\dot{y}_1 = \sin(\ln(\rho))x_1 + \cos(\ln(\rho))\dot{x}_1 - \cos(\ln(\rho))x_2 + \sin(\ln(\rho))\dot{x}_2$$

using that $\dot{x}_1 = -x_1 + x_2$ and $\dot{x}_2 = -x_1 - x_2$ we get

$$\dot{y}_1 = -\cos(\ln(\rho))x_1 - \sin(\ln(\rho))x_2 = -y_1$$

Analogously

$$\dot{y}_2 = \cos(\ln(\rho))x_1 - \sin(\ln(\rho))\dot{x}_1 + \sin(\ln(\rho))x_2 + \cos(\ln(\rho))\dot{x}_2$$

or

$$\dot{y}_2 = \sin(\ln(\rho))x_1 - \cos(\ln(\rho))x_2 = -y_2$$

This implies that

$$\dot{Y} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} Y.$$

(c) (10 points (bonus)) Write the conjugacy between

$$\dot{X} = AX \tag{5}$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and (4). Here $\alpha < 0$. (**Hint**: First modify H of part (b) to conjugate (5) to the system with matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$.)

Solution: Let $H_\beta(x_1, x_2) = (z_1, z_2)$ be defined by

$$\begin{cases} z_1 &= \cos(\beta \ln(\rho))x_1 + \sin(\beta \ln(\rho))x_2 \\ z_2 &= -\sin(\beta \ln(\rho))x_1 + \cos(\beta \ln(\rho))x_2 \end{cases}$$

then we have

$$\dot{Z} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} Z.$$

This follows from a computation almost identical to that of point (b). Let now $G_\alpha(z_1, z_2) = (y_1, y_2)$ be defined by

$$\begin{cases} y_1 &= \operatorname{sgn}(z_1)|z_1|^{-\frac{1}{\alpha}} \\ y_2 &= \operatorname{sgn}(z_2)|z_2|^{-\frac{1}{\alpha}} \end{cases}$$

then

$$\dot{Y} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} Y.$$

Finally $I_{\alpha,\beta} = G_\alpha \circ H_\beta$ is the conjugacy we were looking for.

Question 4 .... *10 point*

Let A be a matrix such that

$$A^2 = I$$

where I is the identity matrix. Show that

$$e^{tA} = \cosh(t)I + \sinh(t)A.$$

(**Hint:** You need the power series expansion of $\cosh(t)$ and $\sinh(t)$. To find them you can use that $\cosh(t) = \cos(it)$ and $\sinh(t) = -i\sin(it)$.)

Solution: First we find that:

$$\cosh(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (it)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}$$

while

$$\sinh(t) = i\sum_{n=0}^{\infty} \frac{(-1)^n (it)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}$$

Observe now that $A^{2n} = I$ while $A^{2n+1} = A$ so that

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + A\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = \cosh(t)I + \sinh(t)A.$$