Name: ____________________

Question:	1	2	3	Total
Points:	30	20	40	90
Score:				

Question 1 .... *30 point*

Given 3 continuous random variable T_1, T_2 and T_3 with j.p.d.f given by $f(t_1, t_2, t_3)$ we can define the marginal of T_1

$$f_{T_1}(t_1) = \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_3 f(t_1, t_2, t_3)$$

and analogously for the marginals on T_2 and T_3. Let now the j.p.d.f. of T_1, T_2 and T_3 be:

$$f(t_1, t_2, t_3) = \begin{cases} \lambda^3 e^{-\lambda t_3} & \text{if } t_3 > t_2 > t_1 > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) (10 points) Compute the marginals $f_{T_1}(t_1)$, $f_{T_2}(t_2)$, $f_{T_3}(t_3)$.

Solution:

$$f_{T_1}(t_1) = \int_0^{\infty} dt_2 \int_{t_2}^{\infty} dt_3 \lambda^3 e^{-\lambda t_3} = \int_0^{\infty} dt_2 \lambda^2 e^{-\lambda t_2} = \lambda e^{-\lambda t_1}$$

$$f_{T_2}(t_2) = \int_0^{t_2} dt_1 \int_{t_2}^{\infty} dt_3 \lambda^3 e^{-\lambda t_3} = \int_0^{t_2} dt_1 \lambda^2 e^{-\lambda t_2} = \lambda^2 t_2 e^{-\lambda t_2}$$

$$f_{T_3}(t_3) = \int_0^{t_3} dt_1 \int_{t_2}^{t_3} dt_3 \lambda^3 e^{-\lambda t_3} = \lambda^3 e^{-\lambda t_3} \int_0^{t_3} dt_1 \int_{t_2}^{t_3} dt_3 = \frac{\lambda^3 t_3^2}{2} e^{-\lambda t_3}$$

(b) (10 points) Compute $E(T_1)$, $E(T_2)$ and $E(T_3)$.

Solution:

$$E(T_1) = \int_0^\infty t_1 \lambda e^{-\lambda t_1} dt_1 = \frac{1}{\lambda}$$

$$E(T_2) = \int_0^\infty t_2 \lambda^2 t_2 e^{-\lambda t_2} dt_2 = \frac{2}{\lambda}$$

$$E(T_2) = \int_0^\infty t_3 \frac{\lambda^3 t_3^2}{2} e^{-\lambda t_3} dt_3 = \frac{3}{\lambda}$$

(c) (10 points) Compute the probability that $T_3 > T_1 + T_2$.

Solution:

$$\begin{aligned} P &= \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \int_{t_1+t_2}^\infty dt_3 \lambda^3 e^{-\lambda t_3} = \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \lambda^2 e^{-\lambda(t_1+t_2)} = \\ &= \int_0^\infty dt_1 \lambda e^{-2\lambda t_1} = \frac{1}{2} \end{aligned}$$

Question 2 . . . . *20 point*

Let X_1 and X_2 be two independent continuous r.v. uniformly distributed in $[-1, 1]$. Let $Y = X_1 + X_2$.

(a) (10 points) Compute the $P(Y \leq y)$, that is the probability that $X_1 + X_2 \leq y$, for a given y. (**Hint**: draw the x_1, x_2 plane with the region where the j.p.d.f. of X_1 and X_2 is not 0 and the region where $x_1 + x_2 \leq y$.)

Solution: The j.p.d.f. of X_1 and $_2$ is:

$$f(x_1, x_2) = \begin{cases} \frac{1}{4} & \text{if } -1 \leq x_1 \leq 1 \text{ and } -1 \leq x_2 \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

If $-2 < y < 0$ we have

$$\begin{aligned} P(Y < y) &= \int_{-1}^{1+y} dx_1 \int_{-1}^{y-x_1} \frac{1}{4} dx_2 = \frac{1}{4} \int_{-1}^{1+y} (1 + y - x_1) dx_1 = \\ &= -\frac{1}{8}(1 + y - x_1)^2 \Big|_{-1}^{1+y} = \frac{1}{8}(2 + y)^2 \end{aligned}$$

Similarly if $0 < y < -2$ we have

$$\begin{aligned} P(Y < y) &= 1 - P(Y > y) = 1 - \int_{-1+y}^{1} dx_1 \int_{y-x_1}^{1} \frac{1}{4} dx_2 = \\ &= 1 - \frac{1}{4} \int_{-1+y}^{1} (1 - y + x_1) dx_1 = \\ &= 1 - \frac{1}{8}(1 - y + x_1)^2 \Big|_{-1+y}^{1} = 1 - \frac{1}{8}(2 - y)^2 \end{aligned}$$

(b) (10 points) Use the previous result to compute the p.d.f. of Y.

Solution: Since the p.d.f. $f(y)$ of Y is the derivative of the c.d.f $F(y) = P(Y < y)$ we have

$$f(y) = \begin{cases} \frac{1}{4}(2 + y) & \text{if } -2 < y < 0 \\ \\ \frac{1}{4}(2 - y) & \text{if } 0 < y < -2 \end{cases}$$

Question 3 . . . . *40 point*

You are shopping in a grocery store with two cashiers. Let N_1 be the number of people in line at the first cashier and N_2 the number of people in line at the second cashier when you arrive at the lines. You know that N_1 can 0, 1 or 2 with equal probabilities. The same thing holds for N_2. Finally N_1 and N_2 are independent. When you arrive at the lines you chose the line with less people. If the two lines have the same number of people you randomly chose one of the two with equal probabilities. Let M_1 and M_2 the number of people on each line after you put yourself on one of them.

(a) (10 points) Compute $P(M_1 = 1 \text{ and } M_2 = 1)$. (**Hint**: which values of N_1 and N_2 give you the situation $M_1 = 1$ and $M_2 = 1$. Think at what can have happened when you arrived at the lines.)

Solution:
If $M_1 = 1$ and $M_2 = 1$ then either you had $N_1 = 1$ and $N_2 = 0$ or $N_1 = 0$ and $N_2 = 1$. Both these possibilities have probability 1/9 so that $P(M_1 = 1 \text{ and } M_2 = 1) = 2/9$.

(b) (10 points) Compute $P(M_1 = 2 \text{ and } M_2 = 1)$. (**Hint** wich values of N_1 and N_2 give you the situation $M_1 = 2$ and $M_2 = 1$. Think at what can have happened when you arrived at the lines.)

Solution:
If $M_1 = 2$ and $M_2 = 1$ then either you had $N_1 = 2$ and $N_2 = 0$ or $N_1 = 1$ and $N_2 = 1$. Both these possibilities have probability 1/9 but in the second case you will have $M_1 = 2$ and $M_2 = 1$ only with probability 1/2. Thus $P(M_1 = 1 \text{ and } M_2 = 1) = 3/18$.

(c) (10 points) Compute the j.p.m.f of M_1 and M_2. Represent it as a table.

Solution: Applying the previous reasoning to all possible results we get

	0	1	2	3
0	0	$\frac{1}{18}$	0	0
1	$\frac{1}{18}$	$\frac{2}{9}$	$\frac{3}{18}$	0
2	0	$\frac{3}{18}$	$\frac{2}{9}$	$\frac{1}{18}$
3	0	0	$\frac{1}{18}$	0

(d) (10 points) Compute $\mathrm{Cov}(M_1, M_2)$ and $\mathrm{Corr}(M_1, M_2)$.

Solution: From the table we have:

$$E(M_1) = E(M_2) = \frac{3}{2} \qquad E(M_1^2) = E(M_2^2) = \frac{49}{18} \qquad E(M_1 M_2) = \frac{44}{18}$$

so that

$$V(M_1) = V(M_2) = \frac{49}{18} - \frac{9}{4} = \frac{17}{36}$$

and

$$\mathrm{Cov}(M_1, M_2) = \frac{44}{18} - \frac{9}{4} = \frac{7}{36} \qquad \mathrm{Corr}(M_1, M_2) = \frac{7}{17}$$