No books or notes allowed. No laptop or wireless devices allowed. Write clearly.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	30	20	20	40	110
Score:					

Question 1 .... *30 point*

The following numbers x_i, $i = 1, \ldots, 17$, represent a sample of size $n = 17$ from a given population.

6.46	6.33	6.93	7.12	7.26
11.12	7.03	6.96	5.48	5.35
6.49	7.87	6.02	6.75	5.67
4.01	5.82			

(a) (10 points) Compute the sample median and fourth spread.

Solution:
After ordering the data you obtain

4.01	5.35	5.48	5.67	5.82
6.02	6.33	6.46	6.49	6.75
6.93	6.96	7.03	7.12	7.26
7.87	11.12			

so that:

$$\tilde{x} = 6.49$$
$$lf = 5.82 \qquad uf = 7.03$$
$$fs = 7.03 - 5.82 = 1.21$$

(b) (10 points) Knowing that $\sum_{i=1}^{17} x_i = 112.67$ and $\sum_{i=1}^{17} x_i^2 = 781.35$ compute the sample mean and variance.

Solution:

$$\bar{x} = \frac{112.67}{17} = 6.63$$
$$s^2 = \frac{1}{16}\left(781.35 - \frac{112.67^2}{17}\right) = 2.16$$

(c) (10 points) After checking for outlier, draw a box plot of the data.

Solution: Observe that $4.01 > 5.82 - 1.5 \cdot 1.21$ so that there are no outlier on the lower part of the sample. On the upper part we have $11.12 > 7.03 + 3 \cdot 1.21$ while $7.87 < 7.03 + 1.5 \cdot 1.21$ so that 11.12 is the only outlier and it is an extreme outlier.

Question 2 .... *20 point*

Let $\mathcal{S}$ be a sample space and $A, B, C \subset \mathcal{S}$ be three mutually independent events.

(a) (10 points) Show that A' and B are independent.(**Hint:** use that $B = (A \cap B) \cup (A' \cap B)$)

Solution:
You need to show that

$$P(A' \cap B) = P(A')P(B)$$

Since $(A \cap B) \cup (A' \cap B) = B$ and $(A \cap B) \cap (A' \cap B) = \emptyset$ we have

$$P(A \cap B) + P(A' \cap B) = P(B)$$

or

$$P(A' \cap B) = P(B) - P(A \cap B)$$

But $P(A \cap B) = P(A)P(B)$ so that we get

$$P(A' \cap B) = P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(A')P(B).$$

(b) (10 points) Show that A and $B \cup C$ are independent.(**Hint:** use that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$)

Solution: You need to show that

$$P(A \cap (B \cup C)) = P(A)P(B \cup C)$$

We have

$$P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$

But $(A \cap B) \cap (A \cap C) = A \cap B \cap C$ so and $P(A \cap B \cap C) = P(A)P(B)P(C)$ so that

$$\begin{aligned} P(A \cap (B \cup C)) &= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) = \\ &= P(A)(P(B) + P(C) - P(B)P(C)) \end{aligned}$$

On the other hand, we have

$$P(B \cup C) = P(B) + P(C) - P(B \cap C) = P(B) + P(C) - P(B)P(C)$$

so that

$$P(A \cap (B \cup C)) = P(A)P(B \cup C)$$

Question 3 .... *20 point*

Let X be a normal random variable with mean 4 and standard deviation 5.

(a) (10 points) Find the average and standard deviation of $Y = 4 - 0.2X$.

Solution:

$E(Y) = 4 - 0.2E(X) = 3.2$ $\qquad$ $V(Y) = 0.2^2 V(X) = 1$ $\qquad$ $\sigma_Y = 1$

(b) (10 points) Find the probability that $3 < X < 7$.

Solution:

$$P(3 < X < 7) = P\left(\frac{3-4}{5} < \frac{X-4}{5} < \frac{7-4}{5}\right) = P\left(-\frac{1}{5} < Z < \frac{3}{5}\right)$$

where Z is standard normal. Thus

$$P(3 < X < 7) = \Phi(0.6) - \Phi(-0.2) = 0.305$$

Question 4 . . . . *40 point*

In a bucket there are 1000 red balls and 2000 blue balls. You extract 1 ball and then flip a fair coin. If the result is Head you reinsert the ball in the bucket. If the result is Tail you keep it. You repeat the above procedure 20 times. Call N the number of balls you have kept at the end of the 20 extractions and R the number of red balls among them.

(a) (10 points) Write the p.m.f of N.

Solution:
N is a Binomial r.v. with parameter 20 and 0.5. Thus

$$P(N = n) = \binom{20}{n} 0.5^n 0.5^{20-n} = \binom{20}{n} 0.5^{20}$$

(b) (10 points) What is the probability that $R = 3$ given that $N = 10$, that is find $P(R = 3 \mid N = 10)$?

Solution: If $N = 10$ then R is an hypergeometric r.v with parameter 10, 1000, 2000 so that

$$P(R = 3 \mid N = 10) = \frac{\binom{1000}{3}\binom{2000}{7}}{\binom{3000}{10}}$$

(c) (10 points) Find an expression for the probability that $R = 3$.(**Hint:** use the above results and the law of total probability.)

Solution: We have

$$P(R=3) = \sum_{n=3}^{20} P(R=3 \mid N=n)P(N=n) = 0.5^{20} \frac{\binom{20}{n}\binom{1000}{3}\binom{2000}{n-3}}{\binom{3000}{n}}$$

(d) (10 points) Observing that $20 << 1000$ and $20 << 2000$, find an approximated p.m.f for R.

Solution: We can assume that each ball, upon extraction, is red with probability $\frac{1000}{3000} = \frac{1}{3}$ and blue with probability $\frac{2000}{3000} = \frac{2}{3}$. Moreover we can assume that different extractions are independent.

After flipping the coin we have 3 possible outcomes: the ball is reinserted (with probability $\frac{1}{2}$), the ball is kept and is blue (with probability $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$) or the ball is kept and is red (with probability $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$).

Thus we can say that R is approximately a binomial r.v. with parameters 20 and $\frac{1}{6}$, that is

$$P(R=r) \simeq \binom{20}{r}\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{20-r}$$