Spring 07
Math 3770

Name: __________________
Final exam
Bonetto

1a	1b	1a	2b	2c	2d	2e	2f
3a	3b	4a	4b	4c	5a	5b	Total

1. Consider the circuit shown in the figure. The system fails as soon as one of the two elements fails. Element A has a life time described by an exponential r.v. T_A with parameter 1 while element B has a life time described by an exponential r.v. T_B with parameter 2. The life times of the two elements are independent.

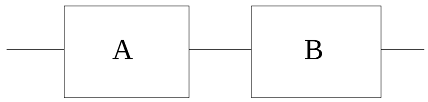

Figure 1: Circuit

(a) let T_S the r.v. that describes the life time of the full system. Compute the p.d.f. of T_S. (**Hint**: Compute $P(T_S > t)$. Use the result to find the c.d.f. of T_S and then its p.d.f.)

If $T_S > t$ it means that both $T_A > t$ and $T_B > t$. Since these two random variable are independent we have $P(T_S > t) = P(T_A > t)P(T_B > t)$. We have that $P(T_A > t) = e^{-t}$ and $P(T_B > t) = e^{-2t}$ so that $P(T_S > t) = e^{-3t}$. It follows that the c.d.f. $F(t)$ of T_S is

$$F(t) = P(T_S < t) = 1 - e^{-3t}$$

and so the p.d.f $f(t)$ of T_S is

$$f(t) = F'(t) = 3e^{-3t}.$$

(b) Suppose that after a time t you observe that the system has failed. Compute the probability that it was element A that failed.

By definition we have to compute

$$P(T_A < t | T_S < t) = \frac{P(T_S < t \& T_A < t)}{P(T_S < t)} = \frac{P(T_A < t)}{P(T_S < t)} = \frac{1 - e^{-t}}{1 - e^{-3t}}.$$

2. The following data x_i are the result of a random sample of size $N = 50$. They are ordered in increasing order.

11.760	14.911	16.267	16.316	16.709	16.756	17.184	17.195	17.459	17.491
17.571	17.621	17.707	18.211	18.260	18.426	18.545	18.729	18.899	19.170
19.251	19.322	19.391	19.500	19.563	19.739	19.752	19.761	20.214	20.230
20.343	20.385	20.549	20.802	21.113	21.323	21.380	21.524	21.704	21.890
21.960	22.073	22.778	22.918	23.418	23.535	24.398	24.650	24.771	26.730

Table 1: Data

You know that

$$\sum_{i=1}^{50} x_i = 990.16 \qquad \sum_{i=1}^{50} x_i^2 = 19983.$$

(a) Compute the sample average and standard deviation.

$$\bar{x} = \frac{990.16}{50} = 19.803 \qquad \sigma_x = \sqrt{\frac{1}{49}\left(19983 - \frac{990.16^2}{50}\right)} = 2.76$$

(b) Compute the median and fourth spread.

$$\tilde{x} = \frac{x_{25} + x_{26}}{2} = 19.651$$

$$lf = x_{13} = 17.707 \qquad uf = x_{38} = 21.524$$

$$f_s = 21.524 - 17.707 = 3.817$$

(c) After finding eventual outlier, draw a box plot for the data.

$$lf - 1.5f_s = 11.982 \qquad lf - 3f_s = 6.2560$$
$$uf + 1.5f_s = 27.250 \qquad uf + 3f_s = 32.975$$

Hence only $x_1 = 11.760$ *is an outlier and there is no extreme outlier.*

(d) Choose a reasonable number of classes and draw an histogram for the data. Show your computation.

Since we have 50 data it is reasonable to choose 7 classes. Each class size is $\delta = 2.14$

	1st class	2nd class	3rd class	4th class	5t class	6th class	7th class
lower boundary	11.760	13.899	16.037	18.176	20.314	22.453	24.591
upper boundaries	13.899	16.037	18.176	20.314	22.453	24.591	26.730
frequency	1	1	11	17	12	7	1
relative frequency	0.02	0.02	0.22	0.34	0.24	0.14	0.02
density	0.0094	0.0094	0.103	0.159	0.112	0.065	0.0094

(e) Give a 98% confidence interval for the true population mean μ.

First we need $z_{0.01} = 2.33$. *Then we have*

$$\mu \in \left[\bar{x} - \frac{2.33\sigma_x}{\sqrt{50}}, \bar{x} + \frac{2.33\sigma_x}{\sqrt{50}} \right] = [18.894, 20.712]$$

(f) How big should the sample be to obtain a precision w smaller than 1. The precision w is the size of the confidence interval.

$$N > \left(\frac{2 \cdot 2.33 \cdot 2.76}{1} \right)^2 = 166$$

3. Suppose now that you have to test the null hypothesis $H_0{:}\mu = 20$ against the alternative hypothesis $H_a{:}\mu < 20$ using the data presented in Table 1.

   (a) Test the hypothesis at 0.025 significance level.

   First we need $z_{0.025} = 1.96$. *The rejection area is thus* $z < -1.96$. *We find that*

   $$z = \frac{\bar{x} - \mu_0}{2.33/\sqrt{50}} = \frac{19.803 - 20}{0.329} = -0.596$$

   so that we do not reject H_0.

   (b) Give the P-value for the test and expalin its meaning.

   $$P = \Phi(-0.596) = 0.276$$

   It means that if the significance level of the test si greater than P *we will reject* H_0 *while if the significance level is less that* P *we will not reject* H_0.

4. The arrival times of the next two cars at a pay booth are described by the two random variables T_1 and T_2. The j.p.d.f. of these variables is:

$$f(t_1, t_2) = \begin{cases} 4e^{-2t_2} & t_2 > t_1 > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal p.d.f. $f_{T_1}(t_1)$ and $f_{T_2}(t_2)$.

$$f_{T_1}(t_1) = 4\int_{t_1}^{\infty} e^{-2t_2}\, dt_2 = -2e^{-2t_2}|_{t_1}^{\infty} = 2e^{-2t_1}$$

$$f_{T_2}(t_2) = 4\int_{0}^{t_2} e^{-2t_2}\, dt_1 = 4t_2 e^{-2t_2}$$

(b) Compute the conditional p.d.f. $f_{T_1|T_2}(t_1|t_2)$ and $f_{T_2|T_1}(t_2|t_1)$.

$$f_{T_1|T_2}(t_1|t_2) = \frac{f(t_1, t_2)}{f_{T_2}(t_2)} = \begin{cases} \frac{1}{t_2} & t_2 > t_1 > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

$$f_{T_2|T_1}(t_2|t_1) = \frac{f(t_1, t_2)}{f_{T_1}(t_1)} = \begin{cases} 2e^{-2(t_2 - t_1)} & t_2 > t_1 > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(c) (**Bonus**) Compute the j.p.d.f. of T_1 and $T_2 - T_1$. Are they independent?

Let $S = T_2 - T_1$ and $g(t_1, s)$ the joint p.d.f. of T_1 and S. We have

$$g_{T_1}(t_1) = f_{T_1}(t_1)$$

and

$$P(T_2 - T_1 > s | T_1 = t_1) = P(T_2 > t_1 + s | T_1 = t_1) = e^{-2s}$$

so that

$$g_{S|T_1}(s|t_1) = 2e^{-2s}$$

This implies that

$$g(t_1, s) = g_{S|T_1}(s|t_1) g_{T_1}(t_1) = 4e^{-2t_1} e^{-2s}.$$

The two r.v. are clearly independent.

5. The data reported in Table 2 are the result of a random sample from a population distributed according to the following p.d.f.:

$$f(x;\alpha) = \begin{cases} \alpha x^{-\alpha-1} & x > 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is a parameter to be determined.

1.9230	1.7780	1.1149	1.3711	1.5444	1.0371	3.4376

Table 2: Data

(a) Use the methods of moment to estimate α.

Let X be the r.v. that describe our population. We have

$$E(X) = \int_1^\infty x\alpha x^{-\alpha-1}dx = \frac{\alpha}{\alpha-1}$$

on the other side we have

$$\bar{x} = \frac{1}{7}\sum_{i=1}^{7} x_i = 1.7437$$

Thus from the methods of moments we have

$$\frac{\alpha}{\alpha-1} = 1.7437 \qquad \alpha = 2.3446$$

(b) Use Maximum Likelihood to estimate α.

The joint p.d.f. of the sample is

$$h(\alpha) = \alpha^7 \prod_{i=1}^{7} x_i^{-\alpha-1} = \alpha^7 e^{-(\alpha+1)\sum_{i=1}^7 \log(x_i)}$$

thus we have

$$h'(\alpha) = 7\alpha^6 e^{-(\alpha+1)\sum_{i=1}^7 \log(x_i)} - \alpha^7 e^{-(\alpha+1)\sum_{i=1}^7 \log(x_i)} \sum_{i=1}^{7} \log(x_i)$$

Observe that $h(0) = h(\infty) = 0$ and $h(\alpha) \geq 0$ so that the maximum is at the solution of $h'(\alpha) = 0$ that is

$$7 - \alpha \sum_{i=1}^{7} \log(x_i) \qquad \alpha = \frac{7}{\sum_{i=1}^7 \log(x_i)}$$

Since $\sum_{i=1}^7 \log(x_i) = 3.3596$ we have

$$\alpha = 2.0836$$