Chapter 5.7 n. 4

Since

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

we have that if $\alpha \le 1$ then $f(x|\alpha,\beta)$ is a decreasing function and the mode ~~of~~ is 0. If $\alpha > 1$ then

$$f'(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left[(\alpha - 1) - \beta x\right] x^{\alpha-2} e^{-\beta x}$$

so that the mode is $\frac{\alpha-1}{\beta}$. Observe that $f(0|\alpha,\beta)=0$ in this case so that it is a minimum for f.

n. 10

Let X_i be the lifetime of the i-th component. The lifetime of the system is

$$Y = \min_{i=1..n} X_i$$

Observe that

$$\mathbb{P}(Y \geq y) = \mathbb{P}(\min_i X_i > y) =$$

$$\mathbb{P}(X_i > y \ \ \forall i) =$$

(independence) $$\mathbb{P}(X_1 > y)^n = (e^{-\mu^{-1} y})^n =$$

$$e^{-n\mu^{-1} y}$$

so that

$$f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \geq y) = n\mu^{-1} e^{-n\mu^{-1} y}$$

and Y is an exponential r.v. with mean $\mathbb{E}(Y) = \frac{\mu}{n}$ and variance $Var(Y) = \frac{\mu^2}{n^2}$

n. 19

Let $Y = X^b$. We know that

$$f_Y(y) = \frac{d}{dy}\left(y^{1/b}\right) f(y^{1/b} \mid a, b) =$$

$$\frac{1}{b} y^{\frac{1}{b}-1} \frac{b}{a^b}\left(y^{1/b}\right)^{b-1} e^{-y/a^b} = \frac{1}{a^b} e^{-y/a^b}$$

Chapter 5.8

n. 5

We need to compute

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{r+\alpha-1}(1-x)^{s+\beta-1}\,dx =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(r+\alpha)\Gamma(s+\beta)}{\Gamma(\alpha+\beta+r+s)} =$$

$$= \frac{\prod_{i=0}^{r-1}(\alpha+i)\prod_{i=0}^{s-1}(\beta+i)}{\prod_{i=0}^{r+s-1}(\alpha+\beta+i)}$$

n. 8

Let A be the event that the selected item is defective. Then we have

$$\mathbb{P}(A \mid X=\theta) = \theta$$

so that

$$\mathbb{P}(A) = \int_0^1 \theta\, f(\theta \mid \alpha,\beta)\, d\theta = \frac{\alpha}{\alpha+\beta}$$

Similarly if $B = \{\text{Two item defective}\}$

$$\mathbb{P}(B \mid X = \theta) = \theta^2$$

$$\mathbb{P}(B) = \int_0^1 \theta^2 f(\theta \mid \alpha, \beta)\, d\theta = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Observe That

$$\mathbb{P}(B) \neq \mathbb{P}(A)^2$$

Chapter 7.2 n 3

The probability of 3 defects according To The prior is:

$$\mathbb{P}(3 \text{ defects}) = 0.4 \cdot \frac{1^3}{3!} e^{-1} + 0.6 \cdot \frac{1.5^3}{3!} e^{-1.5}$$

so that

$$\xi_1(1) = \frac{0.4}{0.4 + 0.6 \cdot 1.5^3 e^{-0.5}}$$

$$\xi_1(1.5) = \frac{0.6 \cdot 1.5^3 e^{-0.5}}{0.4 + 0.6\, 1.5^3 e^{-0.5}}$$

n. 10

We have

$$f(x|\theta) = \begin{cases} 1 & \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$g(\theta) = \begin{cases} \frac{1}{20} & \theta \\ 0 & \text{otherwise} \end{cases}$$

so that

$$f(12|\theta)\, g(\theta) = \begin{cases} \frac{1}{20} & 11.5 \le \theta \le 12.5 \\ 0 & \text{otherwise} \end{cases}$$

and the posterior for θ is uniform in $[11.5, 12.5]$.

Chapter 7.3

n 7

From Theorem 7.3.3 we get that the posterior is normal with

$$\mu_1 = \frac{4 \cdot 68 + 10 \cdot 69.5}{14} = 69.07$$

$$v_1^2 = \frac{4}{14} = \frac{2}{7}$$

n 8

a) Since for a normal p.d.f The mode is equal To The mean and it is decreasing for $x >$ mode and symmetric we have That The interval is

$$[\mu_0 - 0.5, \mu_0 + 0.5] = [67.5, 68.5]$$

b) Anologously The interval is

$$[\mu_1 - 0.5, \mu_1 + 0.5] = [68.57, 69.57]$$

c) For a) we get that, since $v_0^2 = 1$, the probability is

$$1 - 2\Phi(-0.5) = 0.383$$

For b) we have that

$$\frac{0.5}{v_1} = 0.94$$

so that the probability is

$$1 - 2\Phi(-0.94) = 0.652$$

n. 12

Since

$$\frac{\alpha}{\beta} = 0.2$$

$$\frac{\alpha}{\beta^2} = 1$$

we have $\alpha = 0.04$ $\beta = 0.2$.

Thus the posterior in a gamma distribution with $\alpha = 20.04$ and $\beta = 76.2$.