Chapter 3 Exercises

3.8 We have 4 aces, 4 Kings and 44 other cards. Thus

$$p(0,0) = \frac{44 \cdot 43}{52 \cdot 51} = 0.713$$

$$p(0,1) = p(1,0) = 2\,\frac{4}{52} \cdot \frac{44}{51} = 0.132$$

$$p(2,0) = p(0,2) = \frac{4}{52} \cdot \frac{3}{51} = 0.005$$

$$p(1,1) = 2\,\frac{4}{52}\,\frac{4}{51} = 0.012$$

while all other are 0. So we get

Y \ X	0	1	2
0	0.71	0.13	0.005
1	0.13	0.012	0
2	0.005	0	0

3.25 Let ω_1 be a possible value of X and ω_2 a possible value of Y. We have

$$P_{g(X),h(Y)}(\omega_1, \omega_2) = \mathbb{P}(g(X) = \omega_1 \ \& \ h(Y) = \omega_2) =$$

$$= \sum_{x \mid g(x) = \omega_1} \ \sum_{y \mid h(y) = \omega_2} \mathbb{P}(X = x \ \& \ Y = y) \underset{\text{independence}}{=}$$

$$= \sum_{x \mid g(x) = \omega_1} \ \sum_{y \mid h(y) = \omega_2} \mathbb{P}(X = x)\, \mathbb{P}(Y = y) =$$

$$= \sum_{x \mid g(x) = \omega_1} \mathbb{P}(X = x) \sum_{y \mid h(y) = \omega_2} \mathbb{P}(Y = y) =$$

$$P_{g(X)}(\omega_1)\, P_{h(Y)}(\omega_2)$$

3.42 Let 1_{A_i} be the indicator function of A_i. We clearly have

$$N = \sum_i 1_{A_i}$$

so that

$$\mathbb{E}(N) = \mathbb{E}\Big(\sum_i 1_{A_i}\Big) = \sum_i \mathbb{E}(1_{A_i}) = \sum_i \mathbb{P}(A_i)$$

Problems

4 Let's compute

$$\mathbb{P}(U_n \geq u) = \mathbb{P}(\min_i X_i \geq u) =$$

$$= \mathbb{P}(\text{all } X_i \geq u) = \mathbb{P}(X_1 \geq u)^n =$$

$$= \left(\frac{N-u+1}{N}\right)^N$$

so that

$$p_{U_n}(u) = \mathbb{P}(U_n \geq u) - \mathbb{P}(U_n \geq u+1) =$$

$$= \frac{(N-u+1)^n - (N-u)^n}{N^n}$$

Similarly

$$\mathbb{P}(V_n \leq v) = \mathbb{P}(X_1 \leq v)^n = \left(\frac{v}{N}\right)^n$$

and

$$p_{V_n}(v) = \frac{v^n - (v-1)^n}{N^n}$$

7

We have

$$\mathbb{E}(X_1+X_2+\dots+X_N) = \sum_n \mathbb{E}(X_1+X_2+\dots X_N \mid N=n)\cdot \mathbb{P}(N=n) =$$

$$= \sum_n \mathbb{E}(X_1+X_2+\dots+X_n \mid N=n)\,\mathbb{P}(N=n) =$$

$$= \sum_n n\,\mathbb{E}(X_1)\,\mathbb{P}(N=n) = \mu \sum_n n\,\mathbb{P}(N=n) =$$

$$\mu\,\mathbb{E}(N).$$

Chapter 4

Exercises

18 We have

$$G_Y(s) = \sum_y p_Y(y)\, s^y$$

but

$$p_Y(y) \neq 0 \quad \text{iff} \quad p_X(y/k) \neq 0$$

so that

$$G_Y(s) = \sum_x p_X(x)\, s^{kx} =$$

$$\sum_x p_X(x)(s^k)^x = G_X(s^k)$$

Similarly

$$G_Z(s) = \sum_z p_Z(z)\, s^z = \sum_x p_X(x)\, s^{x+k} =$$

$$s^k G_X(s)$$

6.41 We know that

$$G_X(s) = (q+ps)^m$$

while

$$G_Y(s) = (q+ps)^n$$

so that

$$G_{X+Y}(s) = (q+ps)^{n+m}$$

A Bernoulli distribution with par. p is the same thing as a Binomial with parameters p and 1. Thus the sum of n Bernoulli r.v. with par. p is a Binomial r.v with par p and n.

Thus the sum of X and Y is a Binomial with par p and m+n.

Problems

5 Let N be The number of flowers. We have ThaT $N+1$ is a geometric r.v. with par $(1-p)$ so ThaT

$$G_N(s) = s^{-1}\frac{qs}{1-ps} = \frac{q}{1-ps}$$

LeT X_i be the r.v. thaT is 1 if The i-th flower produces a ripe fruit. X_i are Bernoulli with par $\frac{1}{2}$ so thaT

$$G_{X_i}(s) = \frac{1}{2}(1+s)$$

Finally if R is The number of ripe fruit we have

$$R = \sum_{i=1}^{N} X_i$$

so ThaT

$$G_R(s) = \frac{2q}{2-p-ps} = \frac{2q}{2-p} \sum_{r=0}^{\infty} \frac{p^r}{(2-p)^r} s^r .$$

and Thus

$$p_R(r) = \frac{2(1-p)}{2-p} \frac{p^r}{(2-p)^r}$$

For part b we observe That

$$\mathbb{P}(N=n \mid R=r) = \mathbb{P}(R=r \mid N=n) \frac{\mathbb{P}(N=n)}{\mathbb{P}(R=r)}$$

where

$$\mathbb{P}(R=r \mid N=n) = \binom{n}{r} 2^{-n}$$

$$\mathbb{P}(N=n) = (1-p) p^n$$

so That

$$\mathbb{P}(N=n \mid R=r) = \binom{n}{r} 2^{-n-1} (2-p)^{r+1} p^{n-r}$$