This is a take home midterm. You can use your notes, my online notes on canvas and the textbooks book. You are supposed to work on your own text without external help. I'll be available to answer question in person or via email. Please, write clearly and legibly and take a readable scan before uploading.

Name (print): ____________________

Question:	1	2	3	4	5	6	Total
Points:	20	30	10	20	10	10	100
Score:							

Question 1 . . . . *20 point*

(a) (10 points) The number of cars N that arrive at a gas station in an hour is a Poisson r.v. with parameter λ. Each car has a probability p of needing service, independently from all other cars. Find the p.m.f of the number of car Q_s needing service that arrive in an hour. (**Hint:** Write Q_s as a random sum.)

Solution: We can write

$$Q_s = X_1 + X_2 + \cdots + X_N$$

where X_i are independent Bernoulli r.v. with parameter p. Thus we gets

$$G_{Q_s}(s) = G_N(G_X(s)) = e^{\lambda(q+ps-1)} = e^{\lambda p(s-1)}$$

Thus Q_s is a Poisson r.v. with parameter λp and

$$p_{Q_s}(k) = \mathbb{P}(Q_s = k) = e^{-\lambda p}\frac{(\lambda p)^k}{k!}\,.$$

(b) (10 points) Let Q_n be the number of cars not needing service that arrive in an hour. Show that Q_s and Q_n are independent.

Solution: Calling $q = 1-p$ and reasoning as above we get that Q_n is a Poisson r.v. with parameter λq so that

$$p_{Q_n}(k) = e^{-\lambda q}\frac{(\lambda q)^k}{k!}\,.$$

On the other hand, we have

$$\begin{aligned}\mathbb{P}(Q_s =& k_s \,\&\, Q_n = k_n) = \mathbb{P}(Q_s = k_s \,\&\, N = k_s + k_n) = \\ &\mathbb{P}(Q_s = k_s \,|\, N = k_s + k_n)\mathbb{P}(N = k_s + k_n)\,.\end{aligned}$$

Clearly we have

$$\mathbb{P}(Q_s = k_s \,|\, N = n) = \binom{n}{k_s} p^{k_s} q^{n-k_s}$$

so that

$$\mathbb{P}(Q_s = k_s \,\&\, Q_n = k_n) = e^{-\lambda}\lambda^{k_s+k_n}\frac{p^{k_s}q^{k_n}}{k_s!k_n!} = \mathbb{P}(Q_s = k_s)\mathbb{P}(Q_n = k_n)\,.$$

where we have used that $\lambda = \lambda p + \lambda q$.

Question 2 . . . . *30 point*

Let A, B and C be three events. You know that

$$B \perp C, \qquad A \perp (B \cup C), \qquad A \perp (B \cap C), \qquad \text{and} \qquad A \perp (B \cap C^c)$$

where $B \perp C$ means that B and C are independent.

(a) (10 points) Show that

$$\begin{aligned}\mathbb{P}(A \cap B \cap C) =& \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)\\ \mathbb{P}(A \cap B \cap C^c) =& \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C^c)\end{aligned}$$

(**Hint**: for the second one, show first that, for every B and C, if $B \perp C$ then $B \perp C^c$)

Solution: Since $A \perp B \cap C$ we have

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B \cap C)$$

and since $B \perp C$ we get

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

Observe now that if $B \perp C$ then $B \perp C^c$. Indeed

$$\mathbb{P}(B \cap C^c) + \mathbb{P}(B \cap C) = \mathbb{P}(B)$$

so that

$$\mathbb{P}(B \cap C^c) = \mathbb{P}(B) - \mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(B)(1 - \mathbb{P}(C)) = \mathbb{P}(B)\mathbb{P}(B^c)\,.$$

The second identity follow now exactly like the first.

(b) (10 points) Show that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

You can use that $B = (B \cap C) \cup (B \cap C^c)$.

Solution: We have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap ((B \cap C) \cup (B \cap C^c))) = \mathbb{P}((A \cap B \cap C) \cup (A \cap B \cap C^c))$$

Since $(A \cap B \cap C) \cap (A \cap B \cap C^c)) = \emptyset$ we get

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap B \cap C^c)$$

using point (a) we get

$$\begin{aligned}\mathbb{P}(A \cap B) =& \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C^c) =\\ & \mathbb{P}(A)\mathbb{P}(B)(\mathbb{P}(C) + \mathbb{P}(C^c)) = \mathbb{P}(A)\mathbb{P}(B)\end{aligned}$$

(c) (10 points) Show that A, B, and C are independent. (**Hint:** you just miss one independence. Use the hint to point a) to reduce the question to a situation similar to point b).)

Solution: We only need to show that $A \perp C$ or $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$.
Since $A \perp (B \cup C)$ we get $A \perp (B \cup C)^c = (B^c \cap C^c)$. Writing $C^c = (B^c \cap C^c) \cup (B \cap C^c)$ and reasoning as above we get $A \perp C^c$ and thus $A \perp C$.

Question 3 .... *10 point*

You and a friend of yours both toss a fair coin twice. Let X_1 be the number of heads you get and X_2 be the number of heads your friend gets.

Compute the p.m.f. of $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

Solution: We clearly have

$$p_{X_1}(0) = p_{X_1}(2) = \frac{1}{4} \qquad p_{X_1}(1) = \frac{1}{2}$$

and similarly for p_{x_2}.

Thus

$$p_Y(0) = p_{X_1}(0)p_{X_2}(0) = \frac{1}{16}, \quad p_Y(1) = 2p_{X_1}(0)p_{X_2}(1) = \frac{1}{4},$$
$$p_Y(2) = p_{X_1}(1)p_{X_2}(1) + 2p_{X_1}(0)p_{X_2}(2) = \frac{3}{8}.$$
$$p_Y(3) = 2p_{X_1}(2)p_{X_2}(1) = \frac{1}{4}, \quad p_Y(4) = 2p_{X_1}(2)p_{X_2}(1) = \frac{1}{16}$$

On the other call $W = 2 - X_2$. Clearly W is the number of tails your friend gets and has the same p.m.f of X_2. Moreover $Z = X_1 + W - 2$ so that

$$p_Z(z) = p_Y(z + 2).$$

Question 4 .... *20 point*

You roll a fair dye till you get a 6. Let X be the total number of roll and Y be the number of 1 you see in this X rolls.

(a) (10 points) Find $\mathbb{P}(Y = y|X = x)$ and $\mathbb{E}(Y|X = x)$.

Solution: Once X is given all the previous $x-1$ outcomes must be in $\{1, 2, 3, 4, 5\}$. Each of these outcomes is equally probable and they are still independent. Thus

$$\mathbb{P}(Y = y|X = x) = \binom{x-1}{y} \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{x-1-y}$$

and thus

$$\mathbb{E}(Y|X = x) = \frac{x-1}{5}.$$

(b) (10 points) Find $\mathbb{E}(Y)$.

Solution: We have

$$\mathbb{E}(Y) = \sum_{x=1}^{\infty} \mathbb{E}(Y|X = x)\mathbb{P}(X = x) = \sum_{x=1}^{\infty} \frac{x-1}{5}\mathbb{P}(X = x) =$$
$$\frac{1}{5}(\mathbb{E}(X) - 1) = \frac{5}{5} = 1$$

where we have used that X is a geometric r.v. with $p = 1/6$.

Question 5 . . . . *10 point*

A large hall in your building require 100 light bulbs. You receive a shipment of 100 light bulbs and put them on. Each bulb breaks down in the first month of operation with probability $p = 0.1$. If it survives the first month, it breaks down in the second month of operation with probability $p = 0.15$. Suppose that at the end of the second month the number of working bulbs is 80. Obtain the conditional p.m.f. of the number X of bulbs that were working at the end of the first month.

Solution: Clearly the 80 bulbs that are working after two months were working also after one month. For the remaining 20, the fact that they were working after one month are independent events. Let A be the event $\{$*bulb is broken after two mouths*$\}$ and B the event $\{$*bulb was broken after one mouth*$\}$. We need

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}$$

but

$$\mathbb{P}(B) = 0.1 \qquad \mathbb{P}(A) = \mathbb{P}(B) + 0.15\mathbb{P}(B^c) = 0.1 + 0.9 \cdot 0.15 = 0.23$$

so that

$$\mathbb{P}(B|A) = \frac{0.1}{0.1 + 0.9 \cdot 0.15} = 0.43$$

and $X = 80 + Y$ where Y in binomial with parameters 20 and 0.43.

Question 6 .... *10 point*

Look at the following functions and decides whether they can be the c.d.f. of a r.v. In case they are, discuss whether it is a continuous or discrete r.v. and, when possible, write the corresponding p.m.f or p.d.f..

$$F_1(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{3} & 0 \le x < 1 \\ \frac{2}{3} & 1 \le x < 2.5 \\ 1 & x \ge 2.5 \end{cases}$$

$$F_2(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x+1)^2 & -1 \le x < 0 \\ 1 - \frac{1}{2}(x-1)^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F_3(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} + \frac{1}{3}x & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F_4(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2} & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Solution: We have that F_1 is the c.d.f. of a discrete r.v. with p.m.f.

$$p_1(0) = p_1(1) = p_1(2.5) = \frac{1}{3}$$

while F_2 is the c.d.f. of a continuous r.v. with p.d.f.

$$f_2(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \le x < 0 \\ 1-x & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases} .$$

F_3 is a c.d.f. but it describes a r.v. that is neither discrete nor continuous and finally F_4 is not a c.d.f. since it is not right continuous at $x = -1$.