No books or notes allowed. No laptop or wireless devices allowed. Show all your work for full credit. **Write clearly and legibly**.

Name (print): ____________________

Question:	1	2	3	4	Total
Points:	15	25	15	45	100
Score:					

Question 1 .... *15 point*

(a) (15 points) Let X_1 and X_2 be two independent Poisson r.v with parameters μ_1 and μ_2 respectively. Find the p.m.f. of $Y = X_1 + X_2$.

Solution: We have

$$G_{X_i}(s) = e^{\mu_i(1-s)}$$

so that

$$G_Y(s) = G_{X_1}(s)G_{X_2}(s) = e^{(\mu_1+\mu_2)(1-s)}.$$

Thus Y is a Poisson r.v. with parameter $\mu_1 + \mu_2$.

Question 2 . . . . *25 point*

Let A, B abd C be three independent events.

(a) (10 points) Show that A and $B \cap C$ are independent

Solution: You need to show that

$$\mathbb{P}(A \cap (B \cap C)) = \mathbb{P}(A)\mathbb{P}(B \cap C).$$

but $A \cap (B \cap C) = A \cap B \cap C$ so that

$$\mathbb{P}(A \cap (B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

while $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$ so that

$$\mathbb{P}(A)\mathbb{P}(B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

(b) (15 points) Show that A and $B \cup C$ are independent.

Solution: You need to show that

$$\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A)\mathbb{P}(B \cup C).$$

Observe that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ so that

$$\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}((A \cap B) \cap (A \cap C))$$

but $(A \cap B) \cap (A \cap C) = A \cap B \cap C$. Thus

$$\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

On the other hand we have

$$\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C)$$

so that

$$\mathbb{P}(A)\mathbb{P}(B \cup C) = \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

Question 3 . . . . *15 point*

Let X_1 and X_2 be two independent r.v that can take the values -1, 0 and 1. Assume that each of the three value has the same probability for both X_1 and X_2. Compute the p.m.f. of $Y = X_1 + X_2$ and $Z = X_1 X_2$.

Solution: Observe that Y can take the values $-2, -1, 0, 1, 2$. There is one outcome each for which $Y = -2$ or $Y = 2$, two outcomes each for $Y = -1$ or $Y = 1$ and three outcomes for $Y = 0$ so that:

$$p_Y(-2) = p_Y(2) = \frac{1}{9} \qquad p_Y(-1) = p_Y(1) = \frac{2}{9} \qquad p_Y(0) = \frac{1}{3}$$

On the other hand, Z can take the values $-1, 0, 1$. Two outcomes give -1 or 1 and five outcomes give 0. Thus

$$p_Z(-1) = p_Z(1) = \frac{2}{9} \qquad p_Z(0) = \frac{5}{9}$$

Question 4 . . . . *45 point*

A factory produces 1000 computers. Each computer has a probability $p = 0.05$ of having a defect, independently from all other computers. The factory has a quality control department. If a computer is defective it will be detected and discarded with probability 1. If a computer is not defective it will be discarded with a probability $s = 0.03$.

(a) (15 points) Let X be the number of defective computers among the 1000 produced. Write the p.m.f. of X and compute $\mathbb{E}(X)$ and $\mathrm{var}(X)$. This question does not involve the quality control.

Solution: Clearly X is a Binomial r.v. with parameter 1000 and 0.05. It follows that

$$p(x) = P(X = x) = \binom{1000}{x} 0.05^x 0.95^{1000-x}$$

Moreover

$$\mathbb{E}(X) = 1000 \cdot 0.05 = 50 \qquad \mathrm{var}(X) = 1000 \cdot 0.05 \cdot 0.95 = 47.5$$

(b) (10 points) Compute the probability p_1 that a randomly selected computer will be discarded by the quality control department. (**Hint**: Call A the event {*computer is defective*} and B the event {*computer is discarded*}. You are asked to find $p_1 = P(B)$. The text gives you $P(A)$, $P(B|A)$ and $P(B|A')$. Use the Partition Theorem.)

Solution: Call A the event {*computer is defective*} and B the event {*computer is discarded*}. Then we want to find $P(B)$. We have

$$P(B) = P(B|A)P(A) + P(B|A')P(A') = 1 \cdot 0.05 + 0.03 \cdot 0.95 = 0.079$$

(c) (10 points) Compute the probability p_2 that a discarded computer is actually defective. (**Hint**: You are asked to find $p_2 = P(A|B)$. Use Bayes Theorem or the definition of conditional probability.)

Solution: With the notation of the previous point we want $P(A|B)$. We have

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)} = 1\frac{0.05}{0.079} = 0.64$$

(d) (10 points) Call Y the r.v. that describes number of discarded computers and Z the r.v. that describes number of discarded computers that are working. Write the p.m.f. of Y and Z. Are Y and Z independent?

Solution: Observe that at the end of the checking by quality control a computer can be accepted or discarded and this happens independently from any other computer. In the same way it can be discarded and working or not discarded and working (that is not discarded or not working) independently from any other computer. Thus Y and Z are binomial random variables.

Clearly Y is a Binomial r.v. with parameters 1000 and p_1, while Z is a Binomial r.v. with parameters 1000 and p_3 where p_3 is the probability that a randomly selected computer working is both working and discarded. This is given by

$$P(A' \cap B) = P(A')P(B|A') = 0.95 \cdot 0.03 = 0.0285$$

Useful Formulas

- If A and B are events then the probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

and A and B are independent if $P(A|B) = P(A)$.

- If X is a binomial r.v. with parameters N and p then

$$bin(x; N, p) = P(X = x) = \binom{N}{x} p^x (1-p)^{N-x}.$$

Moreover $\mathbb{E}(X) = Np$ and $\mathrm{var}(X) = Np(1-p)$.

- If X is a Poisson r.v. with parameter λ then

$$p(x; \lambda) = P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}.$$

Moreover $\mathbb{E}(X) = \lambda$ and $\mathrm{var}(X) = \lambda$ and $G_X(s) = e^{-\lambda(s-1)}$.