31. Show that $A \sin \omega t + B \cos \omega t = R \sin(\omega t - \delta)$, where $R = \sqrt{A^2 + B^2}$ and δ is the angle defined by $R \cos \delta = A$ and $R \sin \delta = -B$.

32. The field mouse population in Example 3 satisfies the differential equation

$$\frac{dp}{dt} = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.

(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

(c) Find the initial population p_0 if the population is to become extinct in 1 year.

33. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.

(a) Find the rate constant r if the population doubles in 30 days.

(b) Find r if the population doubles in N days.

34. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

(a) If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.

(b) Find an expression for the amount of thorium-234 present at any time t.

(c) Find the time required for the thorium-234 to decay to one-half its original amount.

35. The differential equation for the velocity v of an object of mass m, restricted to vertical motion and subject only to the forces of gravity and air resistance, is

$$m\frac{dv}{dt} = -mg - \gamma v. \qquad \text{(i)}$$

In Eq. (i) we assume that the drag force $-\gamma v$, where $\gamma > 0$ is a drag coefficient, is proportional to the velocity. Acceleration due to gravity is denoted by g. Assume that the upward direction is positive.

(a) Sketch a direction field, including the equilibrium solution, for Eq. (i). Explain the physical significance of the equilibrium solution.

(b) Show that the solution of Eq. (i) subject to the initial condition $v(0) = v_0$ is

$$v = \left(v_0 + \frac{mg}{\gamma}\right) e^{-\gamma t/m} - \frac{mg}{\gamma}.$$

(c) If a ball is initially thrown in the upward direction so that $v_0 > 0$, show that it reaches its maximum height when

$$t = t_{\max} = \frac{m}{\gamma} \ln\left(1 + \frac{\gamma v_0}{mg}\right).$$

(d) The terminal velocity of a baseball dropped from a high tower is measured to be -33 m/s. If the mass of the baseball is 0.145 kg and $g = 9.8$ m/s^2, what is the value of γ?

(e) Using the values for m, g, and γ in part (d), what would be the maximum height attained for a baseball thrown upwards with an initial velocity $v_0 = 30$ m/s from a height of 2 m above the ground?

36. For small, slowly falling objects, the assumption made in Eq. (i) of Problem 35 that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the magnitude of the drag force is proportional to the square of the velocity with the force orientation opposite to that of the velocity.[3]

(a) Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity. Assume that the upward direction is positive.

(b) Determine the limiting velocity after a long time.

(c) If $m = 0.025$ kg, find the drag coefficient so that the limiting velocity is -35 m/s.

37. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g/gal of this chemical flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.

[3]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106*, 2 (1999), pp. 127–135.