No books or notes allowed. No laptop, graphic calculator or wireless devices allowed. Write clearly. Show your work and justify your answers. Remember to add your name to every page.

Name: ______________________________

Question:	1	2	3	4	Total
Points:	26	25	24	25	100
Score:					

1. Evaluate the following definite integrals.

(a) (13 points)

$$\int_0^1 \frac{(x+2)^2-4}{x}dx$$

Solution:

$$\int_0^1 \frac{(x+2)^2-4}{x}dx = \int_0^1 \frac{x^2+4x}{x}dx = \int_0^1 (x+4)dx = \frac{1}{2}+4$$

(b) (13 points)

$$\int_0^1 (x+1)^2\sqrt{3+(x+1)^3}dx$$

Solution: Calling $u = 3+(x+1)^3$ we get $du = 3(x+1)^2dx$. Thus

$$\int_0^1 (x+1)^2\sqrt{3+(x+1)^3}dx = \frac{1}{3}\int_4^{11} \sqrt{u}du = \frac{2}{9}u^{\frac{3}{2}}\Big|_4^{11} = \frac{2}{9}(11\sqrt{11}-8)$$

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2. Evaluate the following indefinite integrals.

(a) (13 points)

$$\int \sin(x)\cos(x)\left(\cos^2(x)-\sin^2(x)\right)dx$$

Solution:

$$\int \sin(x)\cos(x)\left(\cos^2(x)-\sin^2(x)\right)dx = \frac{1}{2}\int \sin(2x)\cos(2x)dx = \frac{1}{8}\sin^2(2x)+C$$

Alternatively call $u=\sin(x)\cos(x)$ so that $du=\left(\cos^2(x)-\sin^2(x)\right)dx$ and

$$\int \sin(x)\cos(x)\left(\cos^2(x)-\sin^2(x)\right)dx = \int udu = \frac{1}{2}(\sin(x)\cos(x))^2+C$$

Observe that

$$(\sin(x)\cos(x))^2 = \frac{1}{4}\sin^2(2x) = \frac{1}{4}-\frac{1}{4}\cos^2(2x) = \frac{1}{4}-\frac{1}{4}(\cos^2(x)-\sin^2(x))^2$$

Since the term $1/4$ can be included in the constant C this gives several different ways to write the result.

(b) (12 points)

$$\int \sin(x)\cos(x)dx$$

Solution: Call $u=\sin(x)$ so that $du=\cos(x)dx$ and

$$\int \sin(x)\cos(x)dx = \int udu = \frac{1}{2}\sin^2(x)+C$$

Equivalently call $u=\cos(x)$ so that $du=-\sin(x)dx$ and

$$\int \sin(x)\cos(x)dx = -\int udu = -\frac{1}{2}\cos^2(x)+C$$

3. Let $F(x)$ be the function defined by:

$$F(x) = \int_{\frac{\pi}{2}}^{x} \frac{\sin(t)}{t} dt.$$

for $x > 0$.

(a) (12 points) Find where F is increasing/decreasing.

Solution: We need to find where $F'(x) \leq 0$ or ≥ 0. We have that

$$F'(x) = \frac{\sin(x)}{x} \tag{1}$$

Since $1/x > 0$ if $x > 0$ we have that $F'(x) \leq 0$ if $x \in [\pi, 2\pi] \cup [3\pi, 4\pi] \cup [5\pi, 6\pi] \cup \ldots$ and $F'(x) \geq 0$ if $x \in (0, \pi] \cup [2\pi, 3\pi] \cup [4\pi, 5\pi] \cup \ldots$ since $F(x)$ is defined for $x > 0$.

Thus we have

$$\begin{array}{lll} F(x) & \text{decreases on} & [\pi, 2\pi],\ [3\pi, 4\pi],\ [5\pi, 6\pi], \ldots \\ F(x) & \text{increases on} & (0, \pi],\ [2\pi, 3\pi],\ [4\pi, 5\pi], \ldots \end{array}$$

(b) (12 points) Is it possible that $F(\pi) > 1$?

Solution: No, it is not possible. Indeed se have that, for $\pi/2 \leq t \leq \pi$:

$$0 \leq \frac{\sin(t)}{t} \leq \frac{1}{t} \leq \frac{2}{\pi}$$

so that

$$F(\pi) = \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(t)}{t} dt \leq \left(\pi - \frac{\pi}{2}\right) \max_{t \in \left[\frac{\pi}{2}, \pi\right]} \frac{\sin(t)}{t} \leq \frac{\pi}{2} \frac{2}{\pi} = 1.$$

Thus we have $F(\pi) \leq 1$.

4. Let A be the region of the plane bounded by the curves

$$f(x) = (x-2)^2 - 1$$

and

$$g(x) = -f(x) = -(x-2)^2 + 1$$

for $1 \leq x \leq 3$.

(a) (12 points) Compute the area of A.

Solution: The area of A is given by:

$$\begin{aligned}\text{Area}(A) &= 2\int_1^3 [-(x-2)^2+1]dx = 2\int_1^3 (-x^2+4x-3)dx = \\ &= 2\left(-\frac{x^3}{3}\Big|_1^3 + 4\frac{x^2}{2}\Big|_1^3 - 3x\Big|_1^3\right) = \frac{8}{3}\end{aligned}$$

(b) (13 points) Let S be the solid obtained by rotating A around the y axis. Compute the volume of S. You may use any of the methods you studied.

Solution: The easiest way is to use Pappus theorem. The centroid of A is at $\bar{x} = 2$ and $\bar{y} = 0$ since A is symmetric with respect to the lines $y = 0$ and $x = 2$. Thus the volume is

$$\text{Vol}(R) = 2\pi\bar{x}\text{Area}(A) = \frac{32\pi}{3}$$

Alternatively you can use the shells method and get

$$\begin{aligned}\text{Vol}(R) &= 4\pi\int_1^3 x[-(x-2)^2+1]dx = 2\int_1^3 (-x^3+4x^2-3x)dx = \\ &= 4\pi\left(-\frac{x^4}{4}\Big|_1^3 + 4\frac{x^3}{3}\Big|_1^3 - 3\frac{x^2}{2}\Big|_1^3\right) = 4\pi\frac{8}{3} = \frac{32\pi}{3}\end{aligned}$$