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Entropic chaoticity for the steady state of a current carrying system

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The steady state for a system of N particles under the influence of an external field and a Gaussian thermostat and colliding with random "virtual" scatterers can be obtained explicitly in the limit of small fields. We show that the sequence of steady state distributions, as N varies, is a chaotic sequence in the sense that the k particle marginal, in the limit of large N, is the k-fold tensor product of the 1 particle marginal. We also show that the chaoticity properties hold in the stronger form of entropic chaoticity. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4824131]

I. INTRODUCTION

Since its introduction by Kac[1] in 1956, the notion of a chaotic sequence has become an important concept in studying many body systems. Chaotic sequences and propagation of chaos are the principal tools for constructing effective equations for many body problems. The aim of this article is to give yet another example of the interplay between chaotic sequences and effective equations.

In Ref. 2, the authors consider a system consisting of N particles moving in a two-dimensional torus and colliding with convex scatterers that form a dispersing billiard. The particles are subject to an external electric field E and a Gaussian thermostat that keeps the kinetic energy fixed. The equation of motion between collisions are given by

$$\begin{cases} \dot{q}_i = v_i & i = 1, \ldots, N \\ \dot{v}_i = \mathbf{F}_i = E - \dfrac{E \cdot j}{U} v_i + F_i \ , \end{cases} \tag{1}$$

where

$$j(\mathbf{V}) = \sum_{i=1}^{N} v_i, \qquad U(\mathbf{V}) = \sum_{i=1}^{N} |v_i|^2 \ , \tag{2}$$

and F_i is the force exerted on the ith particle by collisions with the fixed scatterers. We use the notation $\mathbf{V} = (v_1, v_2, \ldots, v_N)$, where $v_i \in \mathbb{R}^2$, $i = 1, \ldots, N$. Very little is known about billiards with more than one particle. In particular, there is no existence theorem for the Sinai-Ruelle-Bowen (SRB) measure of this system.

The authors introduced in Ref. 2, a stochastic version of the above model in which the deterministic collisions are replaced by Poisson distributed collision processes. More precisely, in the time interval between t and $t + dt$, the ith particle has a probability $|v_i|^\alpha dt$ of suffering a collision. When a collision happens, the velocity of the particle is randomly updated, i.e., if the particle's velocity direction before the collision is $\omega = v/|v|$, after the collision it will be distributed as $K(\omega' \cdot \omega) d\omega'$. The details of the collision kernel K are largely irrelevant. For what follows it will be enough to assume that $K(x) > 0$ for x in an open set $\mathcal{U} \in [-1, 1]$. We note that this stochastic process makes sense for any dimension d. We shall use, as mentioned before, the notation $\mathbf{V} = (v_1, \ldots, v_N)$, $\mathbf{Q} = (q_1, \ldots, q_N)$, with $v_i \in \mathbb{R}^d$ and $q_i \in \mathbb{T}^d$, $i = 1, \ldots, N$.

Let $W(\mathbf{Q}, \mathbf{V}, t; E)$ be the probability density of finding the particles at positions $\mathbf{Q}$ with velocities $\mathbf{V}$ at time t. The time evolution of $W(\mathbf{Q}, \mathbf{V}, t; E)$ is given by the master equation

$$\frac{\partial W(\mathbf{Q},\mathbf{V},t;E)}{\partial t} = -\sum_{i=1}^{N} v_i \cdot \nabla_{q_i} W(\mathbf{Q},\mathbf{V},t;E) - \sum_{i=1}^{N} \nabla_{v_i}\left[\left(E - \frac{E\cdot j(\mathbf{V})}{U(\mathbf{V})} v_i\right) W(\mathbf{Q},\mathbf{V},t;E)\right] + \sum_{i=1}^{N} |v_i|^{\alpha} \int_{\mathcal{S}^{d-1}(1)} K(\omega_i \cdot \omega_i')\Big[W(\mathbf{Q},\mathbf{V}_i',t;E) - W(\mathbf{Q},\mathbf{V},t;E)\Big] d\sigma^{d-1}(\omega'), \tag{3}$$

where $\mathcal{S}^m(R)$ is the m-dimensional sphere with radius R, and $d\sigma^m(\,\cdot\,)$ is the uniform surface measure on $\mathcal{S}^m(R)$. Further, if $\mathbf{V} = (v_1, \dots, v_i, \dots, v_N)$, then $\mathbf{V}_i' = (v_1, \dots, v_i', \dots, v_N)$, with $v_i' = |v_i|\omega_i'$ if $v_i = |v_i|\omega_i$. Note that the variable $\mathbf{Q}$ is not part of the dynamics, i.e., if the initial condition $W(\mathbf{Q}, \mathbf{V}, 0)$ is independent of $\mathbf{Q}$ so will be $W(\mathbf{Q}, \mathbf{V}, t; E)$. Moreover, if $W(\mathbf{Q}, \mathbf{V}, 0)$ is concentrated on the surface of a given energy U_0, that is if

$$W(\mathbf{Q},\mathbf{V},0) = \delta(U(\mathbf{V}) - U_0) F(\mathbf{Q},\mathbf{V},0),$$

then so will be the solution of (3),

$$W(\mathbf{Q},\mathbf{V},t;E) = \delta(U(\mathbf{V}) - U_0) F(\mathbf{Q},\mathbf{V},t;E).$$

Finally, if $F(\mathbf{Q}, \mathbf{V}, 0)$ is a symmetric function so is $F(\mathbf{Q}, \mathbf{V}, t; E)$. Thus, from now on we will only consider symmetric, spatially homogeneous solutions concentrated on the surface of energy $U_0 = N$, that is, on the $dN - 1$ dimensional sphere $\mathcal{S}^{dN-1}(\sqrt{N})$ of radius $\sqrt{N}$. In particular, this means that $F(\mathbf{Q}, \mathbf{V}, t; E) = F(\mathbf{V}, t; E)$ will not depend on the positions $\mathbf{Q}$.

Recall that the k-particle marginal $l_N^{(k)}(v_1, \dots, v_k)$ of a distribution $L_N(\mathbf{V})$ on $\mathcal{S}^{dN-1}(\sqrt{N})$ is defined by the equations

$$\int_{\mathcal{S}^{dN-1}(\sqrt{N})} \varphi(v_1,\dots,v_k) L_N(\mathbf{V}) d\sigma^{dN-1}(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(v_1,\dots,v_k) l_N^{(k)}(v_1,\dots,v_k) dv_1 \cdots dv_k,$$

where $\varphi(v_1, \dots, v_k)$ ranges over the set of bounded continuous function on $\mathbb{R}^{dk}$. Simple computations show that

$$l_N^{(k)}(\mathbf{V}_k) = \sqrt{\frac{N}{N - |\mathbf{V}_k|^2}} \int_{\mathcal{S}^{d(N-k)-1}\left(\sqrt{N-|\mathbf{V}_k|^2}\right)} L_N(\mathbf{V}_k, \mathbf{V}^k) d\sigma^{d(N-k)-1}(\mathbf{V}^k), \tag{4}$$

where $\mathbf{V}_k = (v_1, \dots, v_k)$ and $\mathbf{V}^k = (v_{k+1}, \dots, v_N)$. A sequence of densities $\{L_N\}_{N=1}^{\infty}$ forms a **chaotic sequence with marginal** l if for any bounded continuous function φ,

$$\lim_{N\to\infty} \int_{\mathcal{S}^{dN-1}(\sqrt{N})} \varphi(\mathbf{V}_k) L_N(\mathbf{V}) d\sigma^{dN-1}(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(\mathbf{V}_k) \prod_{j=1}^{k} l(v_j) dv_1 \cdots dv_k. \tag{5}$$

It was shown in Ref. 3 that when $\alpha = 0$ for finite time t the master equation (3) propagates chaos, i.e., the solution of the master equation (3) forms a chaotic sequence if the initial condition does. More precisely, if for any bounded continuous function $\varphi(\mathbf{V}_k)$ the initial condition $F_N(V, 0)$ for the master equation (3) satisfies

$$\lim_{N\to\infty} \int_{\mathcal{S}^{dN-1}} \varphi(\mathbf{V}_k) F_N(\mathbf{V},0) d\sigma(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(\mathbf{V}_k) \prod_{j=1}^{k} f(v_j,0) dv_1 \cdots dv_k,$$

then

$$\lim_{N\to\infty} \int_{\mathcal{S}^{dN-1}} \varphi(\mathbf{V}_k) F_N(\mathbf{V},t;E) d\sigma(\mathbf{V}) = \int_{\mathbb{R}^{dk}} \varphi(\mathbf{V}_k) \prod_{j=1}^{k} f(v_j,t;E) dv_1 \cdots dv_k,$$

where

$$f(v,t;E)=\lim_{N\to\infty} f_N^{(1)}(v,t;E)$$

satisfies the Boltzmann equation

$$\frac{f(v,t;E)}{dt}+\nabla_v\left[\left(E-\frac{E\cdot\hat{j}(t,E)}{u}v\right)f\right]=|v|^\alpha\int_{\mathcal{S}^{d-1}(1)}K(\omega\cdot\omega')\big[f(v',t;E)-f(v,t;E)\big]d\sigma^{d-1}(\omega'). \tag{6}$$

Here, $\hat{j}(t,E)$ and u are given by the self-consistent condition

$$\hat{j}(t,E)=\int vf(v,t;E)dv \qquad \text{and} \qquad u=\int |v|^2 f(v,t;E)dv.$$

It is easy to see that u is independent of time and, since we have chosen $U_0=N$, $u=1$. The initial condition is given by $f(v)=\lim_{N\to\infty} f_N^{(1)}(v,0)$.

Concerning the steady states, the situation is far from clear. In Refs. 2 and 4, it was shown that a steady state $F_{\rm ss}(\mathbf{V};E)$ exists for the master equation (3) provided that $E\neq 0$. If $E=0$ any density $F(\mathbf{V})$ that depends only on the magnitude of the velocities furnishes a stationary state. It is, however, an open question whether $F_{\rm ss}(\mathbf{V};E)$ tends to a limiting distribution as $E\to 0$. Interestingly, assuming that a limiting distribution *exists*, it can be computed exactly and it is given by

$$F_{\rm ss}(\mathbf{V};0)=\delta(U(\mathbf{V})-N)\frac{1}{\widetilde{Z}_N}\frac{1}{\left(\sum_{i=1}^N|v_i|^{2+\alpha}\right)^{\frac{dN-1}{2+\alpha}}}:=\delta(U(\mathbf{V})-N)F_N(\mathbf{V}), \tag{7}$$

where $\widetilde{Z}_N$ is the normalization constant

$$\widetilde{Z}_N=\int_{\mathcal{S}^{dN-1}(\sqrt{N})}\frac{d\sigma^{Nd-1}(\mathbf{V})}{\left(\sum_{i=1}^N|v_i|^{2+\alpha}\right)^{\frac{dN-1}{2+\alpha}}}. \tag{8}$$

For details, the reader should consult Refs. 2 and 4. Thus, the electric field "selects" the right steady state as it tends to zero.

A similar problem exists on the level of the Boltzmann equation. Again it is possible to show that the steady state $f_{\rm ss}(v;E)$ for the Boltzmann equation (6) exists and is unique if $E\neq 0$. This clearly implies the existence of a steady state current $\hat{j}_{\rm ss}(E)$. In Ref. 5 it was shown that, assuming that $f(v)=\lim_{E\to 0} f_{\rm ss}(v,E)$ exists and that $\hat{j}_{\rm ss}(E)=O(E)$, one has

$$f(v)=\frac{\mu^{\frac{d}{2}}}{c}e^{-(\sqrt{\mu}|v|)^{2+\alpha}}, \tag{9}$$

where c and μ are uniquely determined by the normalization of f and the condition $u=1$. One easily computes

$$c:=\int_{\mathbb{R}^d}e^{-|v|^{2+\alpha}}dv=\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\frac{\Gamma\left(\frac{d}{2+\alpha}\right)}{(2+\alpha)} \qquad \text{and} \qquad \mu:=\frac{1}{c}\int_{\mathbb{R}^d}|v|^2e^{-|v|^{2+\alpha}}dv=\frac{\Gamma\left(\frac{d+2}{2+\alpha}\right)}{\Gamma\left(\frac{d}{2+\alpha}\right)}, \tag{10}$$

which for $\alpha=1$ and $d=2$ yield

$$\mu=\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\approx 0.65948 \qquad \text{and} \qquad c=\frac{2\pi}{3}\Gamma\left(\frac{2}{3}\right)\approx 2.83605.$$

For details the reader may consult Ref. 5 where the existence of the small field limit of the steady state distribution is proved for $d=1$.

It is now natural to ask whether the distribution F_N defined in (7) is chaotic with marginal f given by (9). This cannot be deduced from the previous results on propagation of chaos since those results do not hold uniformly in time. A more serious impediment is the fact that the small field limits of the steady states are not known to exist. As explained before, the limit as $E\to 0$ seems to select a steady state for the master equation as well as for its Boltzmann version. It is far from clear that the selection mechanism is such as to preserve chaoticity. In this note we prove that, nevertheless, the distribution defined in (7) is chaotic with marginal (9).

Theorem 1.1. *Let $f_N^{(1)}(v)$ be the one particle marginal of $F_N(\mathbf{V})$ defined in (4) and set*

$$f(v) = \frac{\mu^{\frac{d}{2}}}{c} e^{-(\sqrt{\mu}|v|)^{2+\alpha}} \tag{11}$$

with the constants given by (10). Then for any bounded continuous function $\varphi(v)$,

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} \varphi(v) f_N^{(1)}(v) dv = \int_{\mathbb{R}^d} \varphi(v) f(v) dv \tag{12}$$

and for every k, the k particle marginal $f_N^{(k)}(v_1, \ldots, v_k)$ of $F_N(\mathbf{V})$ satisfies

$$\lim_{N\to\infty} \int_{\mathbb{R}^{kd}} \varphi(v_1, \ldots, v_k) f_N^{(k)}(v_1, \ldots, v_k) dv_1 \cdots dv_k = \int_{\mathbb{R}^{kd}} \varphi(v_1, \ldots, v_k) \prod_{i=1}^{k} f(v_k) dv_1 \cdots dv_k, \tag{13}$$

where, again, φ is any bounded continuous function on $\mathbb{R}^{kd}$. Thus $F_N(\mathbf{V})$ from a chaotic sequence with marginal f.

II. PROOF OF THEOREM 1.1

The following elementary lemma sets the stage for the proof. It will be expressed in terms of the probability distribution

$$g(w) := \frac{e^{-|w|^{2+\alpha}}}{\int_{\mathbb{R}^d} e^{-|w|^{2+\alpha}} dw}.$$

In addition to the constants c and μ given by (10) we need

$$\sigma^2 := \int_{\mathbb{R}^d} (|w|^2 - \mu^2)^2 g(w) dw = \frac{\Gamma\left(\frac{d+4}{2+\alpha}\right)}{\Gamma\left(\frac{d}{2+\alpha}\right)} - \frac{\Gamma\left(\frac{d+2}{2+\alpha}\right)^2}{\Gamma\left(\frac{d}{2+\alpha}\right)^2}. \tag{14}$$

Lemma 2.1. The following formulas hold for $F_N(\mathbf{V})$:

$$F_N(\mathbf{V}) = \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} \int_0^\infty t^{dN-1} \prod_{j=1}^{N} g(v_j t) \frac{dt}{t}, \tag{15}$$

$$Z_N = \frac{(2+\alpha)}{\Gamma\left(\frac{dN-1}{2+\alpha}\right)} \int_{\mathbb{R}^{dN}} \frac{\prod_{i=1}^N g(w_i)}{|\mathbf{W}|} d\mathbf{W}, \tag{16}$$

and

$$f_N^{(k)}(\mathbf{V}_k) = \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N} \sqrt{\frac{N}{(N-|\mathbf{V}_k|^2)^{dk+1}}} \times \int_{\mathbb{R}^{d(N-k)}} \prod_{j=1}^{k} g\left(\frac{v_j |\mathbf{W}^k|}{\sqrt{N-|\mathbf{V}_k|^2}}\right) |\mathbf{W}^k|^{dk-1} \prod_{j=k+1}^{N} g(w_j) d\mathbf{W}^k. \tag{17}$$

Proof. Formula (15) follows from (7) and

$$A^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty s^\gamma e^{-As} \frac{ds}{s}, \tag{18}$$

valid for all $A > 0$ and $\gamma > 0$ by setting

$$A = \left(\sum_{i=1}^{N} |v_i|^{2+\alpha}\right), \ \gamma = \frac{dN-1}{2+\alpha}$$

and substituting $s = t^{2+\alpha}$. The normalization constant $\widetilde{Z}_N$, given in (8), is then

$$\widetilde{Z}_N = \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \int_{\mathcal{S}^{Nd-1}(\sqrt{N})} \int_0^\infty t^{dN-1} \prod_{j=1}^{N} e^{-\left(|v_j|t\right)^{2+\alpha}} \frac{dt}{t} d\sigma^{Nd-1}(\mathbf{V})$$

which, using Fubini's theorem and changing variables $v_j = \sqrt{N} w_j$ equals

$$\frac{(2+\alpha)N^{\frac{Nd-1}{2}}}{\Gamma(\frac{dN-1}{2+\alpha})} \int_0^\infty \int_{\mathcal{S}^{Nd-1}(1)} \prod_{j=1}^{N} e^{-(|w_j|t\sqrt{N})^{2+\alpha}} d\sigma^{Nd-1}(\mathbf{W}) t^{dN-1} \frac{dt}{t}.$$

One more variable change $r = \sqrt{N} t$ yields

$$\frac{(2+\alpha)}{\Gamma(\frac{dN-1}{2+\alpha})} \int_0^\infty \int_{\mathcal{S}^{Nd-1}(1)} \prod_{j=1}^{N} e^{-(|w_j|r)^{2+\alpha}} d\sigma^{Nd-1}(\mathbf{W}) r^{dN-1} \frac{dr}{r}.$$

Taking into account the normalization in the definition of $g(w)$, one obtains

$$Z_N = \frac{(2+\alpha)}{\Gamma(\frac{dN-1}{2+\alpha})} \int_0^\infty \int_{\mathcal{S}^{Nd-1}(1)} \prod_{j=1}^{N} g(w_j r) d\sigma^{Nd-1}(\mathbf{W}) r^{dN-1} \frac{dr}{r},$$

which is the integral (16) written in terms of polar coordinates.

To see (17) we start with (4) and find

$$f_N^{(k)}(\mathbf{V}_k) = \sqrt{\frac{N}{N-|\mathbf{V}_k|^2}} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} \int_{\mathcal{S}^{d(N-k)-1}(\sqrt{N-|\mathbf{V}_k|^2})} \int_0^\infty t^{dN-1} \prod_{j=1}^{N} g(v_j t) \frac{dt}{t} d\sigma^{d(N-k)-1}(\mathbf{V}^k).$$

Once more, using Fubini's theorem and changing variables $v_j = \sqrt{N-|\mathbf{V}_k|^2} w_j$, $j = k+1, \ldots, N$ yields

$$\sqrt{\frac{N}{N-|\mathbf{V}_k|^2}} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} (N-|\mathbf{V}_k|^2)^{\frac{d(N-k)-1}{2}} \times$$

$$\int_0^\infty t^{dN-1} \prod_{j=1}^{k} g(v_j t) \int_{\mathcal{S}^{d(N-k)-1}(1)} \prod_{j=k+1}^{N} g\left(\sqrt{N-|\mathbf{V}_k|^2} w_j t\right) d\sigma^{d(N-k)-1}(\mathbf{W}^k) \frac{dt}{t}.$$

Changing variables $r = \sqrt{N-|\mathbf{V}_k|^2} t$ yields the expression

$$\sqrt{\frac{N}{N-|\mathbf{V}_k|^2}} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})} \frac{1}{Z_N} \frac{1}{(N-|\mathbf{V}_k|^2)^{\frac{dk}{2}}} \times$$

$$\int_0^\infty r^{d(N-k)-1} \prod_{j=1}^{k} g\left(\frac{v_j r}{\sqrt{N-|\mathbf{V}_k|^2}}\right) r^{dk-1} \int_{\mathcal{S}^{d(N-k)-1}(1)} \prod_{j=k+1}^{N} g\left(w_j r\right) d\sigma^{d(N-k)-1}(\mathbf{W}^k) dr,$$

which equals

$$\frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N} \sqrt{\frac{N}{(N-|\mathbf{V}_k|^2)^{dk+1}}} \int_{\mathbb{R}^{d(N-k)}} \prod_{j=1}^{k} g\left(\frac{v_j |\mathbf{W}^k|}{\sqrt{N-|\mathbf{V}_k|^2}}\right) |\mathbf{W}^k|^{dk-1} \prod_{j=k+1}^{N} g\left(w_j\right) d\mathbf{W}^k.$$

□

The following elementary lemma will be used to reduce the computation of the large N limit of (15)–(17) to the law of large numbers.

Lemma 2.2. Let p be a probability distribution on $\mathbb{R}^d$ bounded by some constant C and let $0 \le a < d$. Then

$$\int_{\mathbb{R}^d} \frac{p(y)}{|y|^a} dy \le \frac{d}{d-a} \left(\frac{C|\mathcal{S}^{d-1}|}{d} \right)^{\frac{a}{d}} . \tag{19}$$

Proof. The bathtub principle (see, e.g., Ref. 6) states that the maximum of the expression $\int_{\mathbb{R}^d} \frac{p(y)}{|y|^\alpha} dy$ over all probability distributions p with $p(y) \le C$ is attained at

$$p^*(y) = \begin{cases} C \text{ if } |y| \le R \\ 0 \text{ if } |y| > R, \end{cases}$$

with

$$R = \left(\frac{d}{C|\mathcal{S}^{d-1}|} \right)^{\frac{1}{d}} .$$

The result follows from a straightforward computation. □

The following serves to demonstrate our simple method with the least amount of fuss.

Lemma 2.3. Let a be a positive constant and p be a probability distribution bounded by C with finite second and fourth moments. Set $m := \int_{\mathbb{R}^d} p(y)|y|^2 dy$. Then

$$\lim_{N\to\infty} \int_{\mathbb{R}^{Nd}} \left(\frac{\sqrt{N}}{|\mathbf{W}|} \right)^a \prod_{j=1}^N p(w_i) dw_i = \left(\frac{1}{\sqrt{m}} \right)^a . \tag{20}$$

Proof. We denote

$$\mathbb{P}(A) := \int_A \prod_{j=1}^N p(w_j) d\mathbf{W}.$$

Define the set

$$A_\varepsilon := \left\{ \mathbf{W} \in \mathbb{R}^{Nd} : \left| \frac{|\mathbf{W}|^2}{N} - m \right| \le \varepsilon \right\},$$

so that

$$\left(\frac{1}{\sqrt{m+\varepsilon}} \right)^a \mathbb{P}(A_\varepsilon) < \int_{A_\varepsilon} \left(\frac{\sqrt{N}}{|\mathbf{W}|} \right)^a \prod_{j=1}^N p(w_i) dw_i < \left(\frac{1}{\sqrt{m-\varepsilon}} \right)^a \mathbb{P}(A_\varepsilon).$$

Chebyshev's inequality states that

$$\mathbb{P}(A_\varepsilon^c) \le \frac{s^2}{\varepsilon^2 N}, \tag{21}$$

where

$$s^2 := \int_{\mathbb{R}^d} p(y)(|y|^2 - m^2)^2 dy,$$

so that

$$\left(\frac{1}{\sqrt{m+\varepsilon}} \right)^a \left(1 - \frac{s^2}{\varepsilon^2 N} \right) < \int_{A_\varepsilon} \left(\frac{\sqrt{N}}{|\mathbf{W}|} \right)^a \prod_{j=1}^N p(w_i) dw_i < \left(\frac{1}{\sqrt{m-\varepsilon}} \right)^a \left(1 + \frac{s^2}{\varepsilon^2 N} \right).$$

It remains to estimate

$$\int_{A_\varepsilon^c} \left(\frac{\sqrt{N}}{|\mathbf{W}|} \right)^a \prod_{j=1}^N p(w_i) dw_i \ .$$

By the inequality between the arithmetic and geometric mean

$$\frac{\sqrt{N}}{|\mathbf{W}|} \leq \prod_{j=1}^{N} |w_j|^{-\frac{1}{N}}$$

and hence

$$\int_{A_\varepsilon^c} \left(\frac{\sqrt{N}}{|\mathbf{W}|}\right)^a \prod_{j=1}^{N} p(w_i)dw_i \leq \int_{A_\varepsilon^c} \prod_{j=1}^{N} p(w_i)|w_i|^{-\frac{a}{N}}dw_i = \int_{A_\varepsilon^c} \prod_{j=1}^{N} \gamma_N(w_i)dw_i \left(\int_{\mathbb{R}^d} p(w)|w|^{-\frac{a}{N}}dw\right)^N, \tag{22}$$

where

$$\gamma_N(w) := \frac{p(w)|w|^{-\frac{a}{N}}}{\int_{\mathbb{R}^d} p(w)|w|^{-\frac{a}{N}}dw}$$

is a probability measure. It is easy to see that

$$m_N := \int_{\mathbb{R}^d} \gamma_N(w)|w|^2 dw$$

converges to m as $N \to \infty$ and hence $m - \varepsilon/2 \leq m_N \leq m + \varepsilon/2$ for all N large enough. Likewise, the fourth moment

$$s_N^2 := \int_{\mathbb{R}^d} \gamma_N(w)(|w|^2 - m_N^2)^2 dw$$

converges to σ^2 as $N \to \infty$. Thus we have that the set

$$B_\varepsilon := \left\{\mathbf{W} \in \mathbb{R}^{Nd} : \left|\frac{|\mathbf{W}|^2}{N} - m_N\right| \leq \frac{\varepsilon}{2}\right\} \subset A_\varepsilon$$

and hence $A_\varepsilon^c \subset B_\varepsilon^c$ for all N sufficiently large. Applying Chebyshev's inequality (21) to the measure $\prod_{j=1}^{N} \gamma(v_j)dv_j$ we find that

$$\int_{A_\varepsilon^c} \prod_{j=1}^{N} \gamma_N(w_i)dw_i \leq \int_{B_\varepsilon^c} \prod_{j=1}^{N} \gamma_N(w_i)dw_i \leq \frac{4s_N^2}{\varepsilon^2 N}.$$

Finally, using Lemma 2.2 with a replaced by $\frac{a}{N}$, we get

$$\int_{A_\varepsilon^c} \left(\frac{\sqrt{N}}{|\mathbf{W}|}\right)^a \prod_{j=1}^{N} p(w_i)dw_i \leq \frac{4s_N^2}{\varepsilon^2 N}\left(\frac{d}{d-\frac{a}{N}}\right)^N \left(\frac{C|\mathcal{S}^{d-1}|}{d}\right)^{\frac{a}{d}} \leq \frac{4s_N^2}{\varepsilon^2 N}\left(\frac{Ce|\mathcal{S}^{d-1}|}{d}\right)^{\frac{a}{d}},$$

which tends to zero as $N \to \infty$. Note that we have used the fact that $(1 - \frac{c}{N})^N$ is monotonically decreasing in N. □

Corollary 2.1. We have the following limit:

$$\lim_{N\to\infty} \int_{\mathbb{R}^{Nd}} \frac{\sqrt{N}}{|\mathbf{W}|} \prod_{j=1}^{N} g(w_j)d\mathbf{W} = \frac{1}{\sqrt{\mu}},$$

so that

$$\lim_{N\to\infty} \sqrt{N}\Gamma\left(\frac{dN-1}{2+\alpha}\right) Z_N = \frac{(2+\alpha)}{\sqrt{\mu}}.$$

We now turn our attention to $f_N^{(k)}$. According to (17), we have to compute

$$\int d\mathbf{V}_k \varphi(\mathbf{V}_k) f_N^{(k)}(\mathbf{V}_k) =$$

$$= \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha})Z_N} \int d\mathbf{V}_k \varphi(\mathbf{V}_k) \sqrt{\frac{N}{(N-|\mathbf{V}_k|^2)^{dk+1}}}$$

$$\times \int_{\mathbb{R}^{d(N-k)}} \prod_{j=1}^{k} g\left(\frac{v_j |\mathbf{W}^k|}{\sqrt{N-|\mathbf{V}_k|^2}}\right) |\mathbf{W}^k|^{dk-1} \prod_{j=k+1}^{N} g(w_j) d\mathbf{W}^k.$$

The following change of variables will be helpful.

Lemma 2.4. Let $\mathbf{Y} = (\mathbf{Y}_k, \mathbf{Y}^k)$ *be defined by*

$$\begin{cases} \mathbf{Y}_k = \frac{|\mathbf{W}^k|}{\sqrt{N-|\mathbf{V}_k|^2}} \mathbf{V}_k \\ \mathbf{Y}^k = \mathbf{W}^k \end{cases} .$$

Then

$$\begin{cases} \mathbf{V}_k = \frac{\sqrt{N}}{|\mathbf{Y}|} \mathbf{Y}_k \\ \mathbf{W}^k = \mathbf{Y}^k \end{cases} , \tag{23}$$

and the Jacobian determinant is given by

$$\left| \frac{\partial(\mathbf{V}_k, \mathbf{W}^k)}{\partial(\mathbf{Y})} \right| = \left(\frac{\sqrt{N}}{|\mathbf{Y}|} \right)^{dk} \left(\frac{|\mathbf{Y}^k|^2}{|\mathbf{Y}|^2} \right).$$

Proof. We have

$$|\mathbf{Y}_k|^2 = \frac{|\mathbf{W}^k|^2 |\mathbf{V}_k|^2}{N-|\mathbf{V}_k|^2} = \frac{|\mathbf{Y}^k|^2 |\mathbf{V}_k|^2}{N-|\mathbf{V}_k|^2},$$

so that

$$|\mathbf{V}_k|^2 = \frac{N|\mathbf{Y}^k|^2}{|\mathbf{Y}|^2}$$

from which (23) follows. The Jacobi matrix is of the form

$$\begin{bmatrix} A & B \\ 0 & I_{d(N-k)} \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and the matrices A, B are given by

$$A = \frac{\partial \mathbf{V}_k}{\partial \mathbf{Y}_k} = \frac{\sqrt{N}}{|\mathbf{Y}|} \left(I_{dk} - \frac{\mathbf{Y}_k \otimes \mathbf{Y}_k}{|\mathbf{Y}|^2} \right),$$

$$B = \frac{\partial \mathbf{V}_k}{\partial \mathbf{Y}^k} = -\frac{\sqrt{N}}{|\mathbf{Y}|} \frac{\mathbf{Y}_k \otimes \mathbf{Y}^k}{|\mathbf{Y}|^2}.$$

Note that A is a $dk \times dk$ matrix, and B is a $dk \times d(N-k)$ matrix. Hence, the determinant of the Jacobian is given by $\det A \cdot \det I_{d(N-k)} = \det A$. Because

$$A\mathbf{Y}_k = \frac{\sqrt{N}|\mathbf{Y}^k|^2}{|\mathbf{Y}|^3} \mathbf{Y}_k,$$

$\frac{\sqrt{N}|\mathbf{Y}^k|^2}{|\mathbf{Y}|^3}$ is a simple eigenvalue, and $\frac{\sqrt{N}}{|\mathbf{Y}|}$ is a $(dk - 1)$-fold eigenvalue of A. We thus find that

$$\det A = \left(\frac{\sqrt{N}}{|\mathbf{Y}|}\right)^{dk} \frac{|\mathbf{Y}^k|^2}{|\mathbf{Y}|^2}.$$

□

Let $\varphi(\mathbf{V}_k)$ be a continuous function on $\mathbb{R}^{dk}$ such that

$$\sup_{\mathbf{V}_k \in \mathbb{R}^{dk}} \varphi(\mathbf{V}_k) < K.$$

With the change of variables of Lemma 2.4, we get

$$\begin{aligned}\int d\mathbf{V}_k \varphi(\mathbf{V}_k) f_N^{(k)}(\mathbf{V}_k) &= \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N} \int_{\mathbb{R}^{dk}} d\mathbf{Y}_k \prod_{i=1}^{k} g(y_i) \\ &\quad \times \int_{\mathbb{R}^{d(N-k)}} d\mathbf{Y}^k \frac{\prod_{j=k+1}^{N} g(y_j)}{|\mathbf{Y}|} \varphi\left(y_1 \frac{\sqrt{N}}{|\mathbf{Y}|}, \ldots, y_k \frac{\sqrt{N}}{|\mathbf{Y}|}\right) \\ &=: \int_{\mathbb{R}^{dk}} dy_1 \cdots dy_k \prod_{i=1}^{k} g(y_i) H_N(y_1, \ldots, y_k), \end{aligned} \tag{24}$$

with

$$H_N(\mathbf{Y}_k) = \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N \sqrt{N}} \int_{\mathbb{R}^{d(N-k)}} d\mathbf{Y}^k \varphi\left(\frac{\sqrt{N}}{|\mathbf{Y}|}\mathbf{Y}_k\right) \prod_{j=k+1}^{N} g(y_j) \frac{\sqrt{N}}{|\mathbf{Y}|}.$$

Lemma 2.5. The function $H_N(\mathbf{Y}_k)$ is bounded on $\mathbb{R}^{kd}$, in fact

$$\begin{aligned}|H_N(\mathbf{Y}_k)| &\le K \left(\frac{N}{N-k}\right)^{\frac{dk+2}{2}} \left(\frac{d}{d - \frac{dk+2}{N-k}}\right)^{N-k} \left(\frac{\|g\|_\infty |\mathcal{S}^{d-1}|}{d}\right)^{\frac{dk+2}{d}} \\ &\le K \left(\frac{N}{N-k}\right)^{\frac{dk+2}{2}} \left(\frac{e\|g\|_\infty |\mathcal{S}^{d-1}|}{d}\right)^{\frac{dk+2}{d}}, \end{aligned} \tag{25}$$

which is bounded uniformly in N for $N > k + 1$. Moreover,

$$\lim_{N\to\infty} H_N(\mathbf{Y}_k) = \left(\frac{1}{\sqrt{\mu}}\right)^{dk} \varphi\left(\frac{\mathbf{Y}_k}{\sqrt{\mu}}\right). \tag{26}$$

Proof. Because

$$\frac{1}{|\mathbf{Y}|} \le \frac{1}{|\mathbf{Y}^k|},$$

we get

$$|H_N(\mathbf{Y}_k)| \le K \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N} \int_{\mathbb{R}^{d(N-k)}} d\mathbf{Y}^k \prod_{j=k+1}^{N} g(y_j) \frac{1}{|\mathbf{Y}^k|} \le K \frac{Z_{N-k} \Gamma\left(\frac{d(N-k)-1}{2+\alpha}\right)}{Z_N \Gamma(\frac{dN-1}{2+\alpha})},$$

which is bounded uniformly in N, in fact the limit as $N \to \infty$ of the last expression is K. The proof of (26) follows, again with a slight modification, from the law of large numbers. First, by Corollary 2.1 we have

$$\lim_{N\to\infty} \frac{2+\alpha}{\Gamma(\frac{dN-1}{2+\alpha}) Z_N \sqrt{N}} = \sqrt{\mu}.$$

We denote

$$\mathbb{P}_k(A) := \int_A \prod_{j=k+1}^{N} g(w_j) d\mathbf{Y}^k.$$

Pick ε small and set

$$A_\varepsilon = \left\{ \mathbf{Y}^k \in \mathbb{R}^{(N-k)d} : \left| \frac{|\mathbf{Y}^k|^2}{N-k} - \mu \right| \leq \varepsilon \right\}.$$

Observe that for $\mathbf{Y}^k \in A_\varepsilon$,

$$\sqrt{\frac{N}{N-k}} \frac{1}{\sqrt{\mu + \varepsilon + \frac{|\mathbf{Y}_k|^2}{N-k}}} \leq \frac{\sqrt{N}}{|\mathbf{Y}|} \leq \sqrt{\frac{N}{N-k}} \frac{1}{\sqrt{\mu - \varepsilon + \frac{|\mathbf{Y}_k|^2}{N-k}}}$$

which, because φ is continuous, implies that for $\mathbf{Y}_k$ fixed,

$$\left| \left(\frac{\sqrt{N}}{|\mathbf{Y}|} \right) \varphi \left(\frac{\sqrt{N}}{|\mathbf{Y}|} \mathbf{Y}_k \right) - \mu^{-\frac{1}{2}} \varphi \left(\frac{\mathbf{Y}_k}{\sqrt{\mu}} \right) \right| = o(\varepsilon)$$

uniformly in $\mathbf{Y}^k \in A_\varepsilon$ for N sufficiently large. Needless to say this estimate is *not* uniform in $\mathbf{Y}_k$, which, however, is immaterial for our considerations. From this it follows readily that

$$\left| \int_{A_\varepsilon} \frac{\sqrt{N}}{|\mathbf{Y}|} \varphi \left(\frac{\sqrt{N}}{|\mathbf{Y}|} \mathbf{Y}_k \right) \prod_{j=k+1}^{N} g(w_j) d\mathbf{Y}^k - \mu^{-\frac{1}{2}} \varphi \left(\frac{\mathbf{Y}_k}{\sqrt{\mu}} \right) \mathbb{P}_k(A_\varepsilon) \right| = o(\varepsilon).$$

Using Chebyshev estimate we get

$$\mathbb{P}_k(A_\varepsilon^c) \leq \frac{\sigma^2}{\varepsilon^2 (N-k)},$$

so that

$$\left| \int_{A_\varepsilon} \frac{\sqrt{N}}{|\mathbf{Y}|} \varphi \left(\mathbf{Y}_k \frac{\sqrt{N}}{|\mathbf{Y}|} \right) \prod_{j=k+1}^{N} g(w_j) d\mathbf{Y}^k - \mu^{-\frac{1}{2}} \varphi \left(\frac{\mathbf{Y}_k}{\sqrt{\mu}} \right) \right| \leq o(\varepsilon) + \mu^{-\frac{1}{2}} K \frac{\sigma^2}{\varepsilon^2 (N-k)}. \tag{27}$$

It remains to estimate

$$\begin{aligned} \left| \int_{A_\varepsilon^c} \frac{\sqrt{N}}{|\mathbf{Y}|} \varphi \left(\mathbf{Y}_k \frac{\sqrt{N}}{|\mathbf{Y}|} \right) \prod_{j=k+1}^{N} g(w_j) d\mathbf{Y}^k \right| &\leq K \int_{A_\varepsilon^c} \frac{\sqrt{N}}{|\mathbf{Y}|} \prod_{j=k+1}^{N} g(w_j) d\mathbf{Y}^k \\ &\leq K \left(\frac{N}{N-k} \right)^{\frac{1}{2}} \int_{A_\varepsilon^c} \left(\frac{\sqrt{N-k}}{|\mathbf{Y}^k|} \right) \prod_{j=k+1}^{N} g(w_j) d\mathbf{Y}^k. \end{aligned}$$

Using the same argument as in the proof of Lemma 2.3 yields the estimate

$$\int_{A_\varepsilon^c} \left(\frac{\sqrt{N-k}}{|\mathbf{Y}^k|} \right) \prod_{j=1}^{N} g(y_i) dy_i \leq \frac{4\sigma_{N-k}^2}{\varepsilon^2 (N-k)} \left(\frac{Ce|\mathcal{S}^{d-1}|}{d} \right)^{\frac{1}{d}}.$$

Choosing $\varepsilon = (N-k)^{-\frac{1}{8}}$ and letting $N \to \infty$ proves the lemma. □

Proof of Theorem 1.1.

$$\int d\mathbf{V}_k \varphi(\mathbf{V}_k) f_N^{(k)}(\mathbf{V}_k) = \int_{\mathbb{R}^{dk}} dy_1 \cdots dy_k \prod_{i=1}^{k} g(y_i) H_N(y_1, \ldots, y_k).$$

By Lemma 2.5, $H_N(\mathbf{Y}_k)$ is bounded uniformly in N and converges pointwise to $\varphi(\mathbf{Y}_k/\sqrt{\mu})$ and hence

$$\lim_{N\to\infty}\int d\mathbf{V}_k\varphi(\mathbf{V}_k)f_N^{(k)}(\mathbf{V}_k) = \int_{\mathbb{R}^{dk}} dy_1\cdots dy_k\prod_{i=1}^k g(y_i)\varphi\left(\frac{\mathbf{Y}_k}{\sqrt{\mu}}\right),$$

by the dominated convergence theorem. The last term equals

$$\int_{\mathbb{R}^{dk}} dy_1\cdots dy_k\prod_{i=1}^k f(y_i)\varphi(\mathbf{Y}_k)$$

with f given by (9). Note that f is a probability distribution and μ, defined by (10), yields that $\int_{\mathbb{R}^d}|y|^2 f(y)dy = 1$. This proves the theorem. □

III. EXTENSION AND REMARKS

It is easy to extend the results of Sec. II in a couple of interesting directions. We first observe that one can give a stronger definition of chaoticity by requiring that given a sequence of normalized functions $L_N(\mathbf{V})$ on $\mathcal{S}^{dN-1}(\sqrt{N})$, the entropy per particle of this sequence converges to the entropy of the one particle marginal. More precisely, if

$$S_N = \int_{\mathcal{S}^{dN-1}(\sqrt{N})} L_N(\mathbf{V})\log L_N(\mathbf{V})d\sigma^{dN-1}(\mathbf{V})$$

is the entropy of the N particles system, then

$$\lim_{N\to\infty}\frac{S_N}{N} = \int_{\mathbb{R}^d} l(v)\log l(v)dv$$

where, as before

$$l(v) = \lim_{N\to\infty} l_N^{(1)}(v).$$

If this is true we say that the sequence H_N is **entropically chaotic**, see Ref. 7.

Corollary 3.1. The sequence $F_N(\mathbf{V})$ defined by (7) is entropically chaotic and

$$\lim_{N\to\infty} N^{-1}\int_{\mathcal{S}^{dN-1}(\sqrt{N})} F_N(\mathbf{V})\log F_N(\mathbf{V})d\sigma^{dN-1}(\mathbf{V}) = \log\left(\frac{\mu^{\frac{d}{2}}}{c}\right) - \frac{d}{2+\alpha} = \int_{\mathbb{R}^d} f(v)\log f(v)dv, \tag{28}$$

where μ and c are defined in (10).

Proof. We will just report here the minor modification to the proof of Lemma 2.5 needed to prove the corollary. We observe that

$$x\log x = \lim_{\delta\to 0}\frac{x^{1+\delta}-x}{\delta}.$$

Applying this to (18), we get

$$A^{-\gamma}\log A^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty s^\gamma\log s^\gamma e^{-As}ds - \gamma\psi(\gamma)A^{-\gamma},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the Digamma function. Following the proof of Lemma 2.1, we find

$$\frac{S_N}{N} = -\frac{\log\tilde{Z}_N}{N} - \frac{dN-1}{(2+\alpha)N}\psi\left(\frac{dN-1}{2+\alpha}\right) + \frac{dN-1}{N}\frac{(2+\alpha)}{\Gamma\left(\frac{dN-1}{2+\alpha}\right)Z_N}\int_{\mathbb{R}^{dN}}\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\prod_{i=1}^N g(w_i)}{|\mathbf{W}|}d\mathbf{W}, \tag{29}$$

where $\tilde{Z}_N = c^N Z_N$. Using Stirling formula we get that

$$\lim_{N\to\infty}\left(\frac{\log \tilde{Z}_N}{N} - \frac{dN-1}{(2+\alpha)N}\psi\left(\frac{dN-1}{2+\alpha}\right)\right) = -\log c - \frac{d}{2+\alpha}.$$

Finally, we need to compute the integral in the last term of (29). This can be done exactly as in Corollary 2.1 after a simple extension of the result in Lemma 2.3. Again, we set

$$A_\varepsilon = \left\{\mathbf{W} : \left|\frac{|\mathbf{W}|^2}{N} - \mu\right| < \varepsilon\right\}.$$

For $x < 1$ the function $|(\log x)/x|$ is increasing so that from the inequality of the arithmetic and geometric mean we get

$$\left|\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\sqrt{N}}{|\mathbf{W}|}\right| \le \frac{\sum_{i=1}^N |\log|w_i||}{N}\prod\frac{1}{|w_i|^{\frac{1}{N}}} \qquad \text{for } \frac{|\mathbf{W}|}{\sqrt{N}} < 1.$$

Proceeding as in the proof of Lemma 2.3, we only need to modify (22) as

$$\int_{A_\varepsilon^c, |\mathbf{W}|<\sqrt{N}} \left|\log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\right|\frac{\sqrt{N}}{|\mathbf{W}|}\prod_{j=1}^N g(w_i)dw_i \le$$

$$\int_{A_\varepsilon^c}\prod_{j=1}^N \gamma_N(w_i)dw_i\left(\int_{\mathbb{R}^d} g(w)|w|^{-\frac{1}{N}}dw\right)^{N-1}\int_{\mathbb{R}^d} g(w)\log(|w|)|w|^{-\frac{1}{N}}dw$$

and observe that

$$\int_{\mathbb{R}^d} g(w)\log(|w|)|w|^{-\frac{1}{N}}dw < C'$$

for some constant C' and N large enough. For $x \ge 1$, the non-negative function $\frac{\log x}{x}$ is bounded by $\frac{1}{e}$. Hence

$$\int_{A_\varepsilon^c, |\mathbf{W}|\ge\sqrt{N}} \log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\sqrt{N}}{|\mathbf{W}|}\prod_{j=1}^N g(w_i)dw_i \le \frac{1}{e}\int_{A_\varepsilon^c}\prod_{j=1}^N g(w_i)dw_i \le \frac{s^2}{e\varepsilon^2 N} \tag{30}$$

using (21). Including the integral over A_ε and setting $\varepsilon = N^{-1/8}$ yields

$$\lim_{N\to\infty}\int_{\mathbb{R}^{dN}} \log\left(\frac{|\mathbf{W}|}{\sqrt{N}}\right)\frac{\sqrt{N}}{|\mathbf{W}|}\prod_{i=1}^N g(w_i)\,d\mathbf{W} = \frac{\log\sqrt{\mu}}{\sqrt{\mu}}.$$

Combining the above computations with (29) we get the first equality in (28). The second equality is immediate. □

Another interesting extension is with regards to the first order correction in E. In Ref. 4, under the assumption that the limit $|E| \to 0$ exists, it was shown that

$$F_{\text{ss}}(\mathbf{V}, E) = \delta(U(\mathbf{V}) - N)\left(F_N(\mathbf{V}) + \sum_{i=1}^N E\cdot c(\omega_i)|v_i|R_N(\mathbf{V}) + o(|E|)\right),$$

where $v_i = |v_i|\omega_i$ and

$$R_N(\mathbf{V}) = \frac{1}{\tilde{Z}_N}\frac{dN-1}{\left(\sum_{i=1}^N |v_i|^{2+\alpha}\right)^{\frac{dN-1}{2+\alpha}+1}} = \frac{1}{|v_1|^{2+\alpha}}v_1\cdot\nabla_{v_1}F_N(\mathbf{V}).$$

Here $c(\omega)$ is the unique solution of

$$[(\mathrm{Id} - \mathcal{K})c](\omega) = \omega,$$

where $\mathcal{K}$ is the convolution operator generated by K, that is,

$$(\mathcal{K}c)(\omega) = \int_{\mathcal{S}^{d-1}(1)} K(\omega\cdot\omega')c(\omega')d\sigma^{d-1}(\omega').$$

Because $-c(-\omega)$ is also a solution if $c(\omega)$ is, we have, by uniqueness, that $c(\omega) = -c(-\omega)$. As a consequence

$$\int_{\mathcal{S}^{d-1}(1)} c(\omega')d\sigma^{d-1}(\omega') = 0.$$

Calling $r_N^{(k)}$ the marginal of R_N, we get that

$$r_N^{(k)}(v_1,\ldots,v_k) = \frac{1}{|v_1|^{2+\alpha}} v_1\cdot\nabla_{v_1} f_N^{(k)}(v_1,\ldots,v_k).$$

It is easy to see, from (24), that we can take the limit for $N\to\infty$ on both side and obtain

$$\lim_{N\to\infty} r_N^{(k)}(v_1,\ldots,v_k) = r(v_1)\prod_{i=2}^{k} f(v_k),$$

where

$$r(v) = \frac{1}{|v_1|^{2+\alpha}} v\cdot\nabla_v f(v) = (2+\alpha)\mu^{\frac{2+\alpha}{2}} f(v).$$

Combining the above results we get that the k particle marginal of $F_{\rm ss}$ is

$$\lim_{N\to\infty} f_{\rm ss}^{(k)}(v_1,\ldots,v_k;E) = \left(1 + (2+\alpha)\mu^{\frac{2+\alpha}{2}}\sum_{i=1}^{k} E\cdot c(\omega_i)|v_i|\right)\prod_{i=1}^{k} f(v_k) + o(|E|). \qquad (31)$$

This is consistent with the results on the Boltzmann equation (6). To solve the steady state equation of (6), one as to make an assumption on the form of $\hat{j}_{\rm ss}(E)$ for small $|E|$. It is natural to assume that

$$\hat{j}_{\rm ss}(E) = \tau\underline{\kappa}E + o(|E|), \qquad (32)$$

where $\underline{\kappa}$ is the conductivity tensor for the system with one particle and energy 1, that is,

$$\underline{\kappa} = \frac{1}{|\mathcal{S}^{d-1}(1)|}\int_{\mathcal{S}^{d-1}(1)} c(\omega)\otimes\omega\, d\sigma^{d-1}(\omega).$$

Under this assumption, one finds that

$$f_{\rm ss}(v,E) = \left(1 + (2+\alpha)\nu^{\frac{2+\alpha}{2}} E\cdot c(\omega)|v|\right)\tilde{f}(v) + o(|E|), \qquad (33)$$

where

$$\tilde{f}(v) = \frac{\nu^{\frac{d}{2}}}{b} e^{-(\sqrt{\nu}|v|)^{2+\alpha}},$$

with ν and b uniquely determined by normalization and (32). One can also see that the average energy of this solution is

$$u = \int_{\mathbb{R}^d} |v^2|\tilde{f}(v)\,dv = \left(\frac{\nu}{\mu}\right)^{\frac{2+\alpha}{2}},$$

so that, requiring $u = 1$ we obtain once more the large N limit of the one particle marginal of $F_{\rm ss}$. Clearly, the first order in E of the k-fold tensor product of (33) yields (31). Note that, if the energy per particle is 1, the above results tell us that the current per particle at small field for a large system is $(2+\alpha)\mu^{\frac{2+\alpha}{2}}$ times the current of the one particle system.

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