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Electric fields on a surface of constant negative curvature

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Abstract. A one-parameter family of time reversible Anosov flows is studied; physically, it describes a particle moving on a surface of constant negative curvature under the action of an electric field (corresponding to an automorphic form) and of a 'thermostatting' force (given by Gauss's least-constraint principle). We show that the flows are dissipative, in the sense that the average volume contraction rate is positive and the Sinai–Ruelle–Bowen measure is singular with respect to the volume: therefore they verify the assumptions for the validity of the continuous time version of Gallavotti–Cohen's fluctuation theorem for the large fluctuations of the average volume contraction rate. If several independent electric fields are considered, it makes sense to ask for the validity of Onsager's reciprocity: we show, by explicitly computing the relevant transport coefficents, that it is indeed obeyed.

1. *Introduction*

1.1. In the last few years much work has been dedicated to the clarification of the relationship between statistical mechanics and dynamical systems theory. Many microscopic models for macroscopic statistical mechanical systems have been introduced and studied both analytically and numerically. One of the main goals is to construct a theory for non-equilibrium statistical mechanics. This means that one typically introduces models describing the motion of particles in an external non-conservative field. A classical example is a particle in a billiard moving under the influence of a constant electric field. To prevent the energy of the system to grow unboundedly a mechanism to subtract it from the system must be devised. In this respect the so-called Gaussian thermostat (see Appendix A1 for more details) has also enjoyed a great deal of popularity because of its use in molecular dynamics simulations. An interesting property of this thermostat is that, although dissipative, it preserves the time-reversal properties of the system to which it is applied. In turn this makes such systems ideal for the checking of the *chaotic hypothesis* introduced in [**16**]†. In [**16**] a large-deviations result (*fluctuation theorem*) is proven for Anosov systems; a mathematical presentation of the proof for Anosov diffeomorphisms is in [**13**], and the extension to flows can be found in [**19**]. The application of the fluctuation theorem, according to the chaotic hypothesis, to some mathematical models simulating dissipative reversible systems provides a parameterless law which has been numerically verified at least in few cases [**4, 10**]. A rigorous proof that the average volume contraction rate is positive (which is essentially the definition of dissipativity [**13**]; see also Appendix A2) is, however, still lacking for models on which numerical simulations are performed. The only case in which there exists such a proof is for the one-particle system studied in [**8**], and, in fact, it seems interesting to provide some smooth examples of dissipative Anosov flows and maps.

A theorem by Ruelle [**22**] proves that the average volume contraction rate is strictly positive for Axiom *A* systems (and also for more general systems) when the Sinai–Ruelle–Bowen (SRB) measure is not absolutely continuous with respect to the volume measure, a result which therefore cannot be applied to the simplest Anosov systems because the latter are Hamiltonian flows or area preserving maps.

In this paper we exhibit a simple example of a reversible topologically transitive Anosov flow, for which we can prove that the average volume contraction rate is positive, so that the fluctuation theorem in [**19**] can be applied; the flow is a perturbation of a geodesic flow on a surface of constant negative curvature. A similar analysis can be performed for the case of perturbations of Arnol'd's cat's map. This more trivial example is discussed in Appendix A5: the discussion also shows that, as expected, generically the perturbation of an area preserving map such as Arnol'd's cat's is dissipative, in the sense of [**13**].

The Anosov flow example arises by imagining that the particle on the surface is electrically charged and moves under the influence of an electric field.

There are natural electric fields that can be defined on surfaces of constant negative curvature. Such fields are the analogues of the constant field that can be defined on a flat torus: they are covariant under the action of the group of movements of the (non-Euclidean) geometry of the surface and locally conservative, hence they can be viewed as *electromotive forces* which tend to establish currents circulating around the g *holes* of the surface, if g is the surface genus (the electric fields have non-zero integrals along the contours encircling the holes). There are g linearly independent of such fields, and they are naturally given by g linearly independent automorphic forms that can be defined on the surface.

A free charge on the surface, subject to such fields, will be accelerated: hence the system that we consider is 'thermostatted' by means of a force that imposes that the motion proceeds at constant kinetic energy (or speed). We impose the constraint via Gauss' least-constraint principle.

The system is thus a 'non-Euclidean' version of the system considered in [**8**]. It is smooth and we prove that for small external fields it is dissipative (see Proposition 1.3 below). This could very likely be achieved by using the techniques of [**8**]: however, the high symmetry of our system allows us to provide a direct simple proof by showing that the

† The content of such a hypothesis is that many particle systems in a non-equilibrium stationary state behave, as far as macroscopical quantities are concerned, as if they were Axiom A systems.

dissipation parameter (i.e. the average volume contraction rate) has non-vanishing second derivatives with respect to the fields' strengths which form a strictly positive defined matrix H (the first derivatives vanish by time-reversal symmetry so that for a small field this is enough to prove the positivity).

The derivatives of the average volume contraction rate are computed using a *Green–Kubo's formula* derived heuristically in [**15**] *and proven* in [**23**] for diffeomorphisms. We think that there is no conceptual difficulty to extend the results in [**23**] to mixing Anosov flows, also using the proof in [**7**] for the decay of correlation functions which covers the case of the geodesic flow we are considering.

In the remaining part of this section we give a mathematical definition of the model described above, and state the main result of the paper. In §2 we introduce a more convenient system of coordinates, by following [**9**], and in §3 the positivity of the average volume contraction rate, at small fields, is shown. It is natural to think that the positivity holds at *all* fields; however, our perturbative method cannot deal with fields that are not small (in general the system is no longer an Anosov system when the perturbation becomes too large). In §4 Onsager's coefficients, which form the entries of a matrix L, are computed and shown to be the entries of the (symmetric) matrix H, so that Onsager's reciprocity relations are explicitly verified.

1.2. Let $\mathbb{C}_+$ be the upper complex semiplane: the most general compact analytic surface of constant negative curvature is $\sum_0 = \mathbb{C}_+/\Gamma$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a hyperbolic Fuchsian group; the surface $\sum_0$ can be identified with a fundamental domain of Γ, with the opposite sides identified modulo Γ (we refer to [**9, 11, 18**] for notations). The system is described by the equations of motion (see [**8**] for the analogous billiard model, and the analogy becomes quite clear if one thinks of the billiard motion as a motion on a surface whose curvature is zero except for a *negative* delta-singularity in correspondence of the boundaries)

$$\begin{cases} \dot{x} = y^2 p_x, \\ \dot{y} = y^2 p_y, \\ \dot{p}_x = E_x - \alpha p_x, \\ \dot{p}_y = -y(p_x^2 + p_y^2) + E_y - \alpha p_y, \end{cases} \tag{1.1}$$

where $z = x + iy \in \sum_0$, and the 'electric field' $\mathbf{E} \equiv (E_x, E_y)$ is given by

$$E_x = \frac{\varepsilon}{2}(\phi_1(z) + \overline{\phi_1(z)}), \quad E_y = \frac{i\varepsilon}{2}(\phi_1(z) + \overline{\phi_1(z)}). \tag{1.2}$$

If $\phi_1(z)$ is an automorphic form of order 1, [**11**], and $\overline{\phi_1(z)}$ is an antiautomorphic form of order 1, which is the complex conjugate of the corresponding automorphic form (there are g of both of them, if g is the genus of the compact surface associated to Γ; see [**21**, §2.1 and §2.5]); this means that, if $h \in \mathrm{PGL}(2,\mathbb{R})$ and $j(z,h) = (h_{12}z + h_{22})$, then

$$\phi_1(z\gamma) = \phi_1(z) j(z,\gamma)^2 \quad \forall \gamma \in \Gamma. \tag{1.3}$$

So the (non-conservative) potential is

$$V(z) = -\frac{\varepsilon}{2}\int_C [\phi_1(z')\,dz' + \overline{\phi_1(z')}\,d\overline{z}'],$$

where $\mathcal{C}$ is a curve in $\sum_0$ linking a (arbitrary) point z_0 to z; the potential is multivalued on $\sum_0$. Note that (1.3), the analyticity of the automorphic forms on the upper complex semiplane and the property $d(z\gamma) = j(z,\gamma)^{-2}\,dz$ ensure that the function (1.2) is covariant and locally conservative (its integral over a small closed path, also crossing the boundary, is vanishing), so motivating the fact that it can be interpreted as an electric field.

In (1.1) the function α will be chosen in such a way that

$$\mathcal{H}_0 = \frac{y^2}{2}(p_x^2 + p_y^2) = \frac{y^2\mathbf{p}^2}{2} \tag{1.4}$$

is a constant of the motion (i.e. we assume that the particle moves under the constraint that the kinetic energy is a constant of the motion and the constraint is realized as an ideal one, that is it generates reactions obeying Gauss's least-constraint principle, [**14, 20**]: see Appendix A1). Then one finds

$$\alpha = \frac{\mathbf{p}\cdot\mathbf{E}}{\mathbf{p}^2} \equiv \frac{p_xE_x + p_yE_y}{p_x^2 + p_y^2}, \tag{1.5}$$

so that the *volume contration rate* (i.e. the divergence of the right-hand side of (1.1), up to the sign; see Appendix A2) is

$$\sigma = \alpha = \frac{\mathbf{p}\cdot\mathbf{E}}{\mathbf{p}^2}. \tag{1.6}$$

Note that the equations of motion (1.1) are reversible: the time-reversal symmetry is obvious, namely $(p_x, p_y) \to (-p_x, -p_y)$ and $(x, y) \to (x, y)$; see Appendix A3 for the (interesting) form that the symmetry takes in the other coordinates that we introduce in §2.

Equations (1.1) describe a geodesic flow on a surface of constant negative curvature, $\sum_0$, subject to the action of a non-conservative electric field and coupled to a Gaussian thermostat which keeps constant the free energy of the system. With the units of (1.4), the curvature is $\kappa = -1$.

The main result of this paper follows.

PROPOSITION 1.3. *The system described by (1.1) and (1.2), with α defined in (1.5), for ε small enough, is a dissipative reversible Anosov system, i.e. the time average of the volume contraction rate σ, see (1.6), is strictly positive for almost all (with respect to the volume measure) initial data. Equivalently, the SRB average of σ is strictly positive.*

Finally the existence of several independent electric fields (if the surface genus is $g \geq 2$) allows us to ask the question: 'are Onsager's reciprocity relations and, more generally, Green–Kubo's formulae verified?' for the appropriately defined thermodynamic fluxes. The answer should be affirmative as discussed informally in [**14, 15**]: this can indeed be easily checked in our case (see §4).

It would be interesting to also prove the dissipativity for a system of N particles moving on the surface $\sum_0$ under the influence of the electric field (1.2) and subject to Gauss' least-constraint principle. Nevertheless, despite the fact that the volume contraction rate assumes a very simple expression such as (1.6) for the one-particle system (1.4) (see Appendix A4), we are not able to extend our methods to cover such a case; see also the end comments in Appendix A4.

2. *A global system of coordinates*

2.1. If $w \in \mathbb{C}_+$ and $h \in \mathrm{PGL}(2,\mathbb{R})$, then we write, following [**18**],

$$wh = \frac{h_{11}w + h_{21}}{h_{12}w + h_{22}}. \tag{2.1}$$

The coordinates (q_1, q_2, p_1, p_2) and the matrix $g \in \mathrm{PGL}(2,\mathbb{R})$ are defined by the transformation [**9**],

$$(x, y, p_x, p_y) \in T^*\mathbb{C}_+ \longrightarrow g = \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{R}), \tag{2.2}$$

given by

$$\begin{gathered} z = x + iy = ig^{-1} = \frac{p_2 + iq_1}{p_1 - iq_2} \\ p_x + ip_y = \frac{i}{2}(\det g)^2 \overline{j(i, g^{-1})}^2 = \frac{i}{2}(p_1 + iq_2)^2, \end{gathered} \tag{2.3}$$

which can be rewritten as

$$\begin{gathered} x = \frac{p_1 p_2 - q_1 q_2}{p_1^2 + q_2^2}, \quad y = \frac{q_1 p_1 + q_2 p_2}{p_1^2 + q_2^2}, \\ p_x = -p_1 q_2, \quad p_y = \tfrac{1}{2}(p_1^2 - q_2^2), \end{gathered} \tag{2.4}$$

by taking into account that one has

$$g = \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g^{-1} = \frac{1}{\det g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix},$$

and that the definition of $j(z, h)$ given before (1.3) implies

$$j(i, g^{-1}) = \frac{1}{\det g}(g_{11} - ig_{12}). \tag{2.5}$$

The transformation (2.2) defined by (2.3) is canonical [**9**, Appendix D], and provides a global system of coordinates on the phase space of the geodesic flow deprived of the points with zero velocity, if the elements in PGL(2,$\mathbb{R}$) are identified modulo Γ [**9**, §4]. In terms of the new variables, the Hamiltonian becomes (see [**9**])

$$\mathcal{H}_0 \equiv \mathcal{H}_0(g) = \frac{(\det g)^2}{8} = \frac{(q_1 p_1 + q_2 p_2)^2}{8}. \tag{2.6}$$

Then, from (1.6) and (1.2), one has

$$\begin{aligned} \sigma &= \frac{\varepsilon/2}{p_x^2 + p_y^2}[p_x(\phi_1(z) + \overline{\phi_1(z)}) + ip_y(\phi_1(z) - \overline{\phi_1(z)})] \\ &= \frac{\varepsilon}{2}\left[\frac{\phi_1(z)}{p_x - ip_y} + \frac{\overline{\phi_1(z)}}{p_x + ip_y}\right] = \varepsilon i \left[\frac{\phi_1(ig^{-1})}{(g_{11} - ig_{12})^2} - \frac{\overline{\phi_1(ig^{-1})}}{(g_{11} + ig_{12})^2}\right] \\ &= \frac{\varepsilon i}{(\det g)^2}\left[\frac{\phi_1(ig^{-1})}{j(i, g^{-1})^2} - \frac{\overline{\phi_1(ig^{-1})}}{j(-i, g^{-1})^2}\right] \end{aligned} \tag{2.7}$$

as $ig^{-1} \equiv z$ and $\overline{j(-i, g^{-1})} = j(i, g^{-1})$.

2.2. If we define the matrix $M(g)$ as

$$M(g) = \begin{pmatrix} -D/4 & -c(g) \\ c(g) & D/4 \end{pmatrix}, \tag{2.8}$$

where we have set $D \equiv \det g = \text{constant}$, and

$$c(g) = \frac{\varepsilon}{2D^2}\left[\frac{\phi_1(ig^{-1})}{j(i,g^{-1})^2} + \frac{\overline{\phi_1(ig^{-1})}}{j(-i,g^{-1})^2}\right], \tag{2.9}$$

with $j(i, g^{-1}) = (\det g)^{-1}(g_{11} - ig_{12}) = D^{-1}(g_{11} - ig_{12})$, the equations of motion can be written as

$$\dot{g} = gM(g) = -\frac{D}{4} g\sigma_z + c(g)g\sigma_y, \tag{2.10}$$

with

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

Then, given a smooth function $F \in L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}))$, one has the evolution law

$$\frac{dF}{dt} = -\frac{D}{4}\,\mathrm{Tr}\left(\frac{\partial F}{\partial g}(g\sigma_z)^T\right) + c(g)\,\mathrm{Tr}\left(\frac{\partial F}{\partial g}(g\sigma_y)^T\right). \tag{2.12}$$

Note that $\sigma \equiv \sigma(g) = -2\varepsilon D^{-2}\,\mathrm{Im}(\phi_1(ig^{-1})/j(i,g^{-1})^2)$, while $c(g) = \varepsilon D^{-2}$ $\mathrm{Re}(\phi_1(ig^{-1})/j(i,g^{-1})^2)$. By taking into account the definition of the functions (see [**9**, §4])

$$E_1(g) \equiv \overline{\phi_1(ig^{-1})j(i,g^{-1})^{-2}} = \overline{\phi_1(ig^{-1})}\,j(-i,g^{-1})^{-2}, \tag{2.13}$$

one has

$$c(g) = \varepsilon D^{-2}\,\mathrm{Re}\,E_1(g), \quad \sigma(g) = 2\varepsilon D^{-2}\,\mathrm{Im}\,E_1(g). \tag{2.14}$$

3. *Positivity of the volume contraction rate*

3.1. Let us denote by $\langle\cdot\rangle$ the average with respect to the SRB measure of the perturbed system and by $\langle\cdot\rangle_0$ the average with respect to the SRB measure of the free system (which is the volume measure).

The derivatives of $\langle\sigma\rangle$ with respect to ε are well defined in the case of diffeomorphisms, [**23**]; as we stated in §1.1, we assume the extension of the proof to the case of flows. Then the solution of the equations of motion can be written as $g(t) = g_0(t) + \mathcal{O}(\varepsilon)$, where $g_0(t) = ge^{-\sigma_z Dt/4}$ is the solution of the equations of motion of the unperturbed system with initial data g, (see (2.10) for $\varepsilon = 0$), and, from (2.7), one has $\sigma(g(t)) = \sigma_1(g_0(t))\varepsilon + \mathcal{O}(\varepsilon^2)$, with $\sigma_1(g_0(t)) = 2D^{-2}\,\mathrm{Im}\,E_1(g_0(t))$. From reversibility and [**23**]†,

† One applies twice [**23**, Theorem 3.1(b)] in order to compute the second derivative of $\langle\sigma\rangle$, and then evaluates it in $\varepsilon = 0$. Reversibility is used to conclude that $\langle\sigma_1(\cdot)\rangle_0 = 0$, so that, in particular, the first derivative is vanishing.

one reads (if S^t denotes the time evolution of the perturbed system, and S_0^t denotes the time evolution of the free system)

$$\begin{aligned}\sigma_+ \equiv \langle\sigma\rangle &= \frac{\varepsilon^2}{2}\int_{-\infty}^{\infty} dt\,[\langle\sigma_1(S_0^t\cdot)\sigma_1(\,\cdot\,)\rangle_0] + \mathcal{O}(\varepsilon^3)\\ &= \frac{\varepsilon^2}{2}\int_{-\infty}^{\infty} dt \int dg\,\overline{\sigma_1(g_0(t))}\sigma_1(g) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{3.1}$$

where dg denotes the SRB measure of the free system in terms of g (which is the normalized Haar measure on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$).

3.2. Because of (3.1), one has

$$\left.\frac{\partial^2\langle\sigma\rangle}{\partial\varepsilon^2}\right|_{\varepsilon=0} = \frac{1}{D^4}\int_{-\infty}^{\infty} dt \int dg\,(\overline{E_1(g_0(t))} - E_1(g_0(t)))(E_1(g) - \overline{E_1(g)}). \tag{3.2}$$

Let U be the unitary representation of $\mathrm{PSL}(2,\mathbb{R})$ on $L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}))$ induced by the action of $\mathrm{PSL}(2,\mathbb{R})$ on the homogeneous space $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$,

$$(U(g)f)(g_1) = f(g_1g), \quad q_1 \in L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})),$$

and $Y^{(1)}$ the U-invariant subspace of $L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}))$ on which the representation acts irreducibly spanned by $U(h)E_1$, with h varying in $\mathrm{PSL}(2,\mathbb{R})$, [**18**]. Since $E_1(g_0(t)) = (U(e^{-\sigma_z Dt/4})E_1)(g)$, so that, if $E_1(g) \in Y^{(1)}$, also $E_1(g_0(t)) \in Y^{(1)}$, then

$$\int dg\,(\overline{E_1(g_0(t))} - E_1(g_0(t)))(E_1(g) - \overline{E_1(g)}) = 2\int dg\,\mathrm{Re}(\overline{E_1(g_0(t))}E_1(g)). \tag{3.3}$$

The representation $(U(g)E_1)(g_0) = E_1(g_0g)$ induces the realization $\hat{U}$ of U on $\hat{Y}^{(1)}$

$$(\hat{U}(g)f_1)(z) = f_1(zg)j(z,g)^{-2} \quad \forall f_1 \in \hat{Y}^{(1)}, \tag{3.4}$$

where $\hat{Y}^{(1)}$ can be realized as the subspace of $L_2(\mathbb{C}_+, dx\,dy)$ consisting in the functions analytic in $x+iy \in \mathbb{C}_+$ [**18**]; therefore the identification of $E_1 \in Y^{(1)}$ as a vector $\hat{E}_1 \in \hat{Y}^{(1)}$ [**18**],

$$E_1(g) \longleftrightarrow M_1(z+i)^{-2}, \tag{3.5}$$

with $M_1 = \sqrt{4/\pi}$ yields

$$E_1(g_0(t)) \longleftrightarrow M_1(z_0(t)+i)^{-2}e^{-Dt/2}, \tag{3.6}$$

where $z_0(t) = ig_0^{-1}(t) = ze^{-\sigma_z Dt/4} = e^{-Dt/2}z$. Therefore

$$\int dg\,(\overline{E_1(g_0(t))}E_1(g)) = \frac{4}{\pi}\int_{-\infty}^{\infty} dx \int_0^{\infty} dy\,\frac{e^{-Dt/2}}{(\overline{z_0}(t)-i)^2}\frac{1}{(z+i)^2}. \tag{3.7}$$

One has

$$\int_{-\infty}^{\infty} dx \int_0^{\infty} dy\,\frac{1}{(\overline{z_0}(t)-i)^2}\frac{1}{(z+i)^2} = \frac{\pi e^{Dt}}{(e^{Dt/2}+1)^2},$$

and so, from (3.2),

$$\left.\frac{\partial^2\langle\sigma\rangle}{\partial\varepsilon^2}\right|_{\varepsilon=0} = \frac{4^2}{D^5}. \tag{3.8}$$

As $\langle\sigma\rangle = 0$ for $\varepsilon = 0$, one has that, for $\varepsilon \neq 0$ small enough, $\sigma_+ \equiv \langle\sigma\rangle > 0$, i.e. the system (1.1) is dissipative. If $\mathcal{H}_0 = 1/2$, i.e. $D = 2$, one has that (3.8) is equal to 1/2.

Therefore, since the system (1.1) is an Anosov flow (as it is a smooth perturbation of an Anosov flow, [**1**]), reversibility and dissipativity imply that the fluctuation theorem for flows (in [**19**, Theorem 3.6]) holds.

4. *Onsager's reciprocity relations*

4.1. In general, if the genus g of the surface is $g \geq 2$, one can introduce M electric fields $\mathbf{E}_1, \ldots, \mathbf{E}_M$ which can be written in terms of the $N = g$ linearly independent automorphic forms of order 1, i.e. $\phi_{11}(z), \ldots, \phi_{1N}(z)$, and their complex conjugated (antiautomorphic forms), i.e. $\overline{\phi_{11}(z)}, \ldots, \overline{\phi_{1N}(z)}$, as

$$\begin{aligned} E_{ix}(z) &= \frac{\varepsilon_i}{2}\sum_{j=1}^{N} e_{ij}(\phi_{1j}(z) + \overline{\phi_{1j}(z)}), \quad i = 1, \ldots, M, \\ E_{iy}(z) &= \frac{i\varepsilon_i}{2}\sum_{j=1}^{N} e_{ij}(\phi_{1j}(z) - \overline{\phi_{1j}(z)}), \quad i = 1, \ldots, M, \end{aligned} \tag{4.1}$$

where $\varepsilon_1, \ldots, \varepsilon_M$ are the fields intensities, and the $M \times N$ matrix e_{ij} is real. Then the resulting electric field is given by the superposition of the M electric fields (4.1), i.e. $\mathbf{E} = \sum_{i=1}^{M} \mathbf{E}_i$, and the volume contraction rate is $\sigma = \mathbf{p}^{-2}(\mathbf{p}\cdot\mathbf{E})$.

4.2. The second derivatives of $\langle\sigma\rangle$ with respect to the fields intensities $\varepsilon_1, \ldots, \varepsilon_M$, computed in $\varepsilon_1 = \cdots = \varepsilon_M = 0$, define a matrix H. We consider also the matrix L given by

$$L_{ij} = \frac{\partial}{\partial\varepsilon_i}\left\langle\frac{\partial\sigma}{\partial\varepsilon_j}\right\rangle\bigg|_{\varepsilon_1=\cdots=\varepsilon_M=0}. \tag{4.2}$$

By repeating the analysis of §2 and §3, and taking into account: (1) the fact that the functions in $L_2(\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}))$ corresponding to the automorphic forms of order 1 through (2.14) are orthogonal to each other, and admit all the same realization (3.4); (2) the results in [**23**]; and (3) the reversibility of the equations of motion, one finds (see [**15**])

$$\begin{aligned} L_{ij} &= \int_{-\infty}^{\infty} dt\,[\langle\sigma_{1i}(S_0^t\cdot)\sigma_{1j}(\,\cdot\,)\rangle_0 - \langle\sigma_{1i}(\,\cdot\,)\rangle_0\langle\sigma_{1j}(\,\cdot\,)\rangle_0] \\ &= \int_{-\infty}^{\infty} dt \int dg\,\overline{\sigma_{1i}(g_0(t))}\sigma_{1j}(g), \end{aligned}$$

where $\sigma_{1i}(g) = 2D^{-2}\sum_{j=1}^{N} e_{ij}\,\mathrm{Im}\,E_{1j}(g)$, so that

$$\begin{aligned} L_{ij} &= \frac{2}{D^4}\int_{-\infty}^{\infty} dt \sum_{n,m=1}^{N} e_{in}e_{jm}\int dg\,\overline{E_n(g_0(t))}E_m(g) \\ &= \frac{4^2}{D^5}\sum_{n=1}^{N} e_{in}e_{jn} = \frac{4^2}{D^5}(ee^T)_{ij}. \end{aligned} \tag{4.3}$$

From (4.2) one immediately sees that $L_{ij} = L_{ji}$, which expresses Onsager's reciprocity relations, and (4.3) provides an explicit formula for Onsager's coefficients. If $D = 2$, as after (3.8) and $M = N$, $e_{ij} = \delta_{ij}$, one finds that $L_{ij} = \delta_{ij}/2$, as written at the end of [**17**].

Note also that L is just the second-derivatives matrix of the average volume contraction rate, i.e. $L = H$, if $H_{ij} = \partial^2\langle\sigma\rangle/\partial\varepsilon_i\partial\varepsilon_j$, so that (4.3) shows that H is positive definite: then the system is dissipative.

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Appendix A1. Gauss' least-constraint principle

A1.1. The system described by the Hamiltonian (1.4) can be obtained (locally) from a free system in the Euclidean space $\mathbb{R}^3$, with the constraint that $x_3 = \varphi(x_1, x_2) = \tanh^{-1}\sqrt{1-(x_1^2+x_2^2)} - \sqrt{1-(x_1^2+x_2^2)}$, $x_1^2 + x_2^2 \le 1$; one can think that this equation describes the pseudosphere, i.e. the surface obtained by rotating about the asymptotic the Beltrami tractrix (so that it has constant negative curvature), [**5**]. Such a system can be described by the Lagrangian

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}{2} - \lambda W(x_1, x_2, x_3), \tag{A1.1}$$

with $\mathbf{x} = (x_1, x_2, x_3)$ and $\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$, for a suitable function $W(x_1, x_2, x_3)$, in the limit $\lambda \to \infty$, [**2**, §17]. More formally the following result holds.

LEMMA A1.2. *If* $\mathbf{x} = \mathbf{X}(q)$, *with* $\mathbf{q} = (q_1, q_2, q_3)$, *is a system of local regular coordinates, well adapted and orthogonal on a surface* $\Sigma \in \mathbb{R}^3$, *i.e. (i)* $q_3 = 0$ *describes the surface* $x_3 = \varphi(x_1, x_2)$, *(ii) the kinetic matrix*

$$G_{nm}(\mathbf{q}) = \sum_{i=1}^{3} \frac{\partial X_i(\mathbf{q})}{\partial q_n} \cdot \frac{\partial X_i(\mathbf{q})}{\partial q_m}, \tag{A1.2}$$

is such that

$$\frac{\partial X_i(\mathbf{q})}{\partial q_n} \cdot \frac{\partial X_i(\mathbf{q})}{\partial q_m} = 0 \quad \text{for } n = 1, 2 \text{ and } m = 3, \tag{A1.3}$$

and

$$\left(\frac{\partial X_i(\mathbf{q})}{\partial q_3}\right)^2\Bigg|_{q_3=0} = \text{constant}, \tag{A1.4}$$

and (iii) the potential $W(\mathbf{X}(\mathbf{q}))$ *is such that*

$$\begin{gathered} W(\mathbf{X}(\mathbf{q})) = \tilde{W}(q_3), \quad \tilde{W}(0) = 0, \\ \frac{\partial \tilde{W}(q_3)}{\partial q_3}\bigg|_{q_3=0} = 0, \quad \frac{\partial^2 \tilde{W}(q_3)}{\partial q_3^2}\bigg|_{q_3=0} > 0, \end{gathered} \tag{A1.5}$$

then the solution of the Euler–Lagrange equations corresponding to the Lagrangian (A1.1), in the limit $\lambda \to \infty$, converge to the solution of the Euler–Lagrange equations corresponding to the Lagrangian

$$\mathcal{L}_0(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} \sum_{n,m=1,2} G_{nm}(\mathbf{q}) \dot{q}_n \dot{q}_m, \tag{A1.6}$$

provided that the initial data are such that $\mathbf{x}$ *is on the surface* $x_3 = \varphi(x_1, x_2)$ *and* $\dot{\mathbf{x}}$ *is tangent to the surface (in terms of* $\mathbf{q}$, $q_3 = 0$ *and* $\dot{q}_3 = 0$*).*

A1.3. Proof of Lemma A1.2. The proof follows from [**12**, §3.8, Proposition 13]: we simply sketch it. If $\mathbf{q}$ is an orthonormal and well adapted system of coordinates, then we can write

$$\mathbf{x} = \mathbf{X}(\mathbf{q}), \quad \dot{\mathbf{x}} = \frac{\partial \mathbf{X}}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}}, \tag{A1.7}$$

so that

$$\begin{aligned}
\dot{x}_i &= \left[\sum_{n=1,2} \frac{\partial X_i}{d q_n} \dot{q}_n \right] + \frac{\partial X_i}{d q_3} \dot{q}_3 \\
\ddot{x}_i &= \left[\sum_{n,m=1,2} \frac{\partial X_i}{\partial q_n q_m} \dot{q}_n \dot{q}_m + \sum_{n=1,2} \frac{\partial X_i}{d q_n} \ddot{q}_n \right] + \left[\frac{\partial^2 X_i}{\partial q_3^2} \dot{q}_3^2 + \frac{\partial X_i}{d q_3} \ddot{q}_3 \right] \\
&\quad + \left[2 \sum_{n=1,2} \frac{\partial^2 X_i}{\partial q_n q_3} \dot{q}_n \dot{q}_3 \right],
\end{aligned} \tag{A1.8}$$

where the mixed terms in the final summation are vanishing when computed in $q_3 = 0$, as the new coordinates are well adapted and orthogonal (see (A1.3) above).

Then, on the surface $q_3 = 0$, if (i) the initial data are taken such that $\dot{q}_3(0) = 0$, and (ii) W satisfies the conditions in (A1.5), one has

$$\sum_{n=1,2} \frac{\partial X_i}{d q_n} \ddot{q}_n = - \sum_{n,m=1,2} \frac{\partial^2 X_i}{\partial q_n q_m} \dot{q}_n \dot{q}_m. \tag{A1.9}$$

Therefore the equations (A1.9) and the definition (A1.2) give

$$\sum_{m=1,2} G_{nm}(\mathbf{q}) \ddot{q}_m = -\frac{1}{2} \sum_{k,m=1,2} \frac{\partial G_{nm}(\mathbf{q})}{\partial q_k} \dot{q}_m \dot{q}_k, \tag{A1.10}$$

which can be rewritten, if one defines the ‘momentum’

$$p_n = \sum_{m=1,2} G_{nm}(\mathbf{q}) \dot{q}_m, \tag{A1.11}$$

as

$$\dot{q}_n = \sum_{m=1,2} (G^{-1}(\mathbf{q}))_{nm} p_m, \quad \dot{p}_n = \frac{1}{2} \sum_{k,m=1,2} \frac{\partial G_{nm}(\mathbf{q})}{\partial q_k} \dot{q}_m \dot{q}_k, \tag{A1.12}$$

which are the Euler–Lagrange equations corresponding to the Lagrangian (A1.6). □

A1.4. Let us apply Gauss' least-constraint principle to the system (A1.1) subjected to the action of an electric field

$$\tilde{\mathbf{E}} = (\tilde{E}_1(x_1, x_2, x_3), \tilde{E}_2(x_1, x_2, x_3), \tilde{E}_3(x_1, x_2, x_3)), \tag{A1.13}$$

and then take the limit $\lambda \to \infty$; the field will be chosen in a suitable way to be fixed later (see (A1.18) below).

One has that the equations of motion become (see [**14**, Appendix A1])

$$\ddot{x}_i = -\frac{\partial W}{\partial x_i} + \tilde{E}_i - \alpha \dot{x}_i, \quad \text{for } i = 1, 2, 3, \tag{A1.14}$$

with $\alpha = (\dot{\mathbf{x}} \cdot \tilde{\mathbf{E}})/\dot{\mathbf{x}}^2$. Note that α has the same form independently of W. When the limit $\lambda \to \infty$ is taken, instead of the equations (A1.19), one obtains

$$\sum_{n=1,2} \frac{\partial X_i}{dq_n} \cdot \ddot{q}_n = -\sum_{n,m=1,2} \frac{\partial^2 X_i}{\partial q_n q_m} \dot{q}_n \dot{q}_m + \tilde{E}_i - \alpha \dot{x}_i. \tag{A1.15}$$

Then (A1.10) are replaced with

$$\sum_{m=1,2} G_{nm}(\mathbf{q}) \ddot{q}_m = -\frac{1}{2} \sum_{k,m=1,2} \frac{\partial G_{nm}(\mathbf{q})}{\partial q_k} \dot{q}_m \dot{q}_k + \sum_{i=1}^{3} \frac{\partial X_i}{\partial q_n} \tilde{E}_i - \alpha \sum_{i=1}^{3} \frac{\partial X_i}{\partial q_n} \dot{x}_i, \tag{A1.16}$$

which can be rewritten, if one takes into account again the definition (A1.11), as

$$\begin{aligned} \dot{q}_n &= \sum_{i=1,2} (G^{-1}(\mathbf{q}))_{nm} p_m \\ \dot{p}_n &= \frac{1}{2} \sum_{k,m=1,2} \frac{\partial G_{nm}(\mathbf{q})}{\partial q_k} \dot{q}_m \dot{q}_k + \sum_{i=1}^{3} \frac{\partial X_i}{\partial q_n} \tilde{E}_i - \alpha p_n. \end{aligned} \tag{A1.17}$$

Then one defines the electric field in (A1.13), in such a way that, on the surface $q_3 = 0$,

$$\begin{cases} E_{q_1} = \sum_{i=1}^{3} \frac{\partial X_i}{\partial q_1} \tilde{E}_i = \frac{\varepsilon}{2} (\phi_1(z) + \overline{\phi_1(z)}), \\ E_{q_2} = \sum_{i=1}^{3} \frac{\partial X_i}{\partial q_2} \tilde{E}_i = i \frac{\varepsilon}{2} (\phi_1(z) - \overline{\phi_1(z)}), \end{cases} \tag{A1.18}$$

where $z = q_1 + iq_2$ and $\phi_1(z)$ is as in §1, so that the equations (1.1) are obtained, with

$$\alpha = \frac{\sum_{n=1,2} E_n \dot{q}_n}{\sum_{n=1,2} \dot{q}_n p_n}. \tag{A1.19}$$

For (1.1) one has $G_{nm}(\mathbf{q}) = q_2^{-2} \delta_{nm}$ on the surface $q_3 = 0$†, so that one can rewrite (A1.19) as

$$\alpha = \frac{\sum_{n=1,2} E_n p_n}{\sum_{n=1,2} p_n^2}, \tag{A1.20}$$

so that also equation (1.5) follows.

† For instance, on the surface $q_3 = 0$, one can set in (A1.7), $x_1(q_1, q_2, 0) = q_2^{-1} \cos q_1$, $X_2(q_1, q_2, 0) = q_2^{-1} \sin q_1$ and $X_3(q_1, q_2, 0) = \tanh^{-1} \sqrt{1 - q_2^{-2}} - \sqrt{1 - q_2^{-2}}$, with $q_2 > 1$ and $|q_1| < \pi$.

Appendix A2. Volume contraction rate for diffeomorphisms and flows

A2.1. Given a C^∞ compact Riemannian manifold M and a C^∞ diffeomorphism $S : M \to M$, the *entropy production* rate (or average volume contraction rate) is defined as

$$e(\mu) = -\int_M \mu(dx) \ln \Lambda(x), \tag{A2.1}$$

where $\Lambda(x)$ is the absolute value of the Jacobian of S at x, computed with respect to the Riemann metric, and μ is a S-invariant probability measure associated with the dynamical system (M, S), [**22**, §1].

If (A2.1) is strictly positive, then one has an increment of entropy for the system, which therefore can be interpreted as a dissipative system.

A2.2. Given a C^∞ flow $S^t : M \to M$, which solves the differential equation $\dot{x} = F(x)$ on M, in the same way as in §A2.1 it is possible to show that the entropy production rate is given by

$$e(\mu) = -\int_M \mu(dx) \frac{d}{dt} \ln \lambda_t(x)\bigg|_{t=0}, \tag{A2.2}$$

where $\lambda_t(x)$ is the Jacobian of the linear map $T_x M \to T_{S^t x} M$, and μ is a S^t-invariant measure associated with the dynamical system (M, S^t). If $S^t : M \to M$ is a topologically transitive Anosov flow (more generally an Axiom A flow), as in §1.1, and μ is the SRB measure, then the average with respect to μ of any smooth function F defined on M is equal to its time average:

$$\int_M \mu(dx) F(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, F(S^t x_0), \tag{A2.3}$$

for μ_0-almost all initial data $x_0 \in M$, if μ_0 is the volume measure, see [**6**, Theorem 5.1].

If one sets

$$\ln J(x) = \frac{d}{dt} \ln \lambda_t(x)\bigg|_{t=0}, \tag{A2.4}$$

then (A2.2) corresponds to the expression appearing in [**19**, Definition 3.1], so that $\sigma_+ \equiv \langle \sigma \rangle = e(\mu)$. From the definition (A2.4) and the properties of the derivative of the determinant, it is straightforward to verify that

$$\ln J(x) = \operatorname{div} \dot{x} \equiv \operatorname{div} F(x), \tag{A2.5}$$

so motivating the definition of the volume contraction rate given in §1.2, from which (1.5) follows for the system (1.1).

Appendix A3. Reversibility of the equations of motion

A3.1. The system (1.1) is reversible: in fact there exists an isometric transformation $\mathcal{I}$ such that $\mathcal{I}^2 = \mathbb{1}$ and $S^t \mathcal{I} = \mathcal{I} S^{-t}$, if S^t denotes the time evolution. Such a transformation is given by

$$\mathcal{I}(x, y, p_x, p_y) = (x, y, -p_x, -p_y), \tag{A3.1}$$

which, in terms of the coordinates introduced in (2.1), becomes

$$\mathcal{I}(q_1, q_2, p_1, p_2) = (-p_2, p_1, -q_2, q_1), \tag{A3.2}$$

that is

$$\mathcal{I}g = -\sigma_y g = \begin{pmatrix} -g_12 & g_{11} \\ -g_{22} & g_{21} \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A3.3}$$

A3.2. One has $\det(\mathcal{I}g) = \det g$, $\phi_1(ig^{-1}) = \phi_1(i(\mathcal{I}g)^{-1})$, $j(i,(\mathcal{I}g)^{-1}) = ij(i,g^{-1})$, $c(\mathcal{I}g) = -c(g)$, $\sigma_y(\mathcal{I}g) = -\sigma_y(g)$, $M(\mathcal{I}g) = M^T(g)$, so that, if $g' = \mathcal{I}g$ and $t' = -t$,

$$\begin{aligned} \frac{dg'}{dt'} &= -gM(g)\sigma_y = -g\mathbb{1}M(g)\sigma_y = g\sigma_y^2 M(g)\sigma_y \\ &= g'(\sigma_y M(g)\sigma_y) = g'M(g'), \end{aligned} \tag{A3.4}$$

which shows the invariance of the equations of motion under the action of the map $\mathcal{I}$.

Appendix A4. Many-particle system

A4.1. Let us consider a system of N particles on $\sum_0$, with the free Hamiltonian

$$\mathcal{H}_0 = \sum_{n=1}^{N} \frac{y_n^2}{2}(p_{nx}^2 + p_{ny}^2) = \sum_{n=1}^{N} \frac{y_n^2 \mathbf{p}_n^2}{2}, \tag{A4.1}$$

subject to the action of the electric field (1.2) and coupled with a Gaussian thermostat. Then, by applying Gauss's least-constraint principle and reasoning as in Appendix A1, one finds that the equations of motion become

$$\begin{cases} \dot{x}_n = y_n^2 p_{nx}, \\ \dot{y}_n = y_n^2 p_{ny}, \\ \dot{p}_{nx} = E_{nx} - \alpha p_{nx}, \\ \dot{p}_{ny} = -y_n(p_{nx}^2 + p_{ny}^2) + E_{ny} - \alpha p_{ny}, \end{cases} \tag{A4.2}$$

where $z_n = x_n + iy_n \in \sum_0$, and the 'electric field' $\mathbf{E}_n \equiv (E_{nx}, E_{ny})$ is given by (1.2) evaluated in $z = z_n$.

In (A4.2) the function α has to be chosen in such a way that (A4.1) is a constant of the motion, that is

$$\alpha = \frac{\sum_{n=1}^N y_n^2 \mathbf{p}_n \cdot \mathbf{E}_n}{\sum_{n=1}^N y_n^2 \mathbf{p}_n^2} \equiv \frac{\sum_{n=1}^N y_n^2 (p_{nx}E_{nx} + p_{ny}E_{ny})}{\sum_{n=1}^N y_n^2 (p_{nx}^2 + p_{ny}^2)}, \tag{A4.3}$$

so that the volume contration rate (i.e. the divergence of the right-hand side of (A4.2), up to the sign) is

$$\sigma = (2N-1)\alpha = (2N-1)\frac{\sum_{n=1}^N y_n^2 \mathbf{p}_n \cdot \mathbf{E}_n}{\sum_{n=1}^N y_n^2 \mathbf{p}_n^2}. \tag{A4.4}$$

A4.2. Then, as for (1.1), we can perform the canonical transformation of coordinates

$$\begin{aligned} z_n = x_n + iy_n = ig_n^{-1} = \frac{p_{n2} + iq_{n1}}{p_{n1} - iq_{n2}} \\ p_{nx} + ip_{ny} = \frac{i}{2}(\det g_n)^2 \overline{j(i, g_n^{-1})}^2 = \frac{i}{2}(p_{n1} + iq_{n2})^2, \end{aligned} \tag{A4.5}$$

with $g_n \in \mathrm{PGL}(2, \mathbb{R})$ for all n.

Then from (A4.4) and (A4.5) one obtains that, in terms of $g_1, \ldots, g_N$,

$$\mathcal{H}_0 \equiv \mathcal{H}_0(g_1, \ldots, g_N) = \frac{\sum_{n=1}^{N} (\det g_n)^2}{8} \equiv \frac{\mathcal{D}^2}{8} = \text{constant}, \tag{A4.6}$$

and, for all n,

$$\begin{aligned} y_n^2 \mathbf{p}_n \cdot \mathbf{E}_n &= \frac{(\det g_n)^2}{(p_{n1}^2 + q_{n2}^2)^2} \left[\frac{i\varepsilon}{2} (p_{n1} + iq_{n2})^2 \varphi_1(z_n) + \text{C.C.} \right] \\ &= \frac{i\varepsilon}{2} \left[\frac{\phi_1(z_n)}{j(i, g_n^{-1})^2} - \text{C.C.} \right], \end{aligned} \tag{A4.7}$$

so that (A4.3) becomes

$$\alpha = \frac{\varepsilon i}{\mathcal{D}^2} \sum_{n=1}^{N} \left[\frac{\phi_1(z_n)}{j(i, g_n^{-1})^2} - \frac{\overline{\phi_1(z_n)}}{j(-i, g_n^{-1})^2} \right] = \frac{2\varepsilon}{\mathcal{D}^2} \sum_{n=1}^{N} \operatorname{Im} E_1(g_n), \tag{A4.8}$$

where, in the last line, we have taken into account the definition (2.14) with g replaced with g_n.

Even if the initial decoupled systems are N Anosov systems, the interacting system is no longer an Anosov system (the available volume is different from the original). This is also reflected in the fact that the representations of the functions $E_1(g_n)$ are realized in disjoint spaces, so that the analysis in §3 cannot be repeated. Therefore, nothing can be concluded with the methods of §3.

Appendix A5. A dissipative Anosov diffeomorphism

A5.1. Let us consider a perturbation of Arnol'd's cat's map [**3**, §13]:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}, \tag{A5.1}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{T}^2$, and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$ is a function with components periodic on $\mathbb{T}^2$.

The volume contraction rate σ is defined as the logarithm of the absolute value of determinant of the Jacobian matrix (see Appendix A2):

$$\sigma = \ln|\det J(\mathbf{x})|, \quad J(\mathbf{x}) = \begin{pmatrix} 1 + \varepsilon\partial f_1(\mathbf{x})/\partial x_1 & 1 + \varepsilon\partial f_1(\mathbf{x})/\partial x_2 \\ 1 + \varepsilon\partial f_2(\mathbf{x})/\partial x_1 & 2 + \varepsilon\partial f_2(\mathbf{x})/\partial x_2 \end{pmatrix}, \tag{A5.2}$$

so that, if: (i) $\langle \cdot \rangle$ and $\langle \cdot \rangle_0$ denote the average with respect to the SRB measure, respectively, of the perturbed system and of the free system (the latter is the volume measure), and (ii) S_0^n denotes the nth iterate of the transformation in (A5.1) for $\varepsilon = 0$, i.e. the nth power of the matrix in (A5.1), then, from [**23**],

$$\left.\frac{\partial^2\langle\sigma\rangle}{\partial\varepsilon^2}\right|_{\varepsilon=0} = \sum_{n=-\infty}^{\infty} \langle\sigma_1(S_0^n\,\cdot\,)\sigma_1(\,\cdot\,)\rangle_0, \tag{A5.3}$$

where $\sigma_1(\mathbf{x})$ is the first derivative of $\sigma(\mathbf{x})$ with respect to ε, computed in $\varepsilon = 0$; one has

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \langle\sigma_1(S_0^n\,\cdot\,)\sigma_1(\,\cdot\,)\rangle_0 &= \sum_{n=-\infty}^{\infty} \int_0^1 dx_1 \int_0^1 dx_2\, \sigma_1(S_0^n\mathbf{x})\sigma_1(\mathbf{x}) \\ &= \sum_{n=-\infty}^{\infty} \sum_{\nu\in\mathbb{Z}^2} \overline{\sigma_{1\nu}}\sigma_{1S_0^n\nu},\end{aligned} \tag{A5.4}$$

where $\sigma_{1\nu}$ is the νth Fourier coefficient of σ_1:

$$\sigma_1(\mathbf{x}) = \sum_{\nu\in\mathbb{Z}^2} \sigma_{1\nu} e^{2\pi i\mathbf{x}\cdot\nu}. \tag{A5.5}$$

The series in (A5.4) is well defined (i.e. convergent), see [**23**]. With fixed ν the other sum runs over all the integers n such that to $S_0^n\nu$ there corresponds another Fourier coefficient of σ_1: in general such a sum will not be empty, so that in general the expression in (A5.4) will be not vanishing (and therefore positive), as a consequence of [**22**]†.

A5.2. As a concrete case we can consider a perturbation as in (A5.1), with

$$f_1(\mathbf{x}) = \frac{1}{2\pi}[\sin(2\pi(x_1 + x_2)) + \sin(2\pi x_1)], \quad f_2(\mathbf{x}) = 0, \tag{A5.6}$$

so that

$$\sigma_1(\mathbf{x}) = 2\cos(2\pi(x_1 + x_2)) + \cos(2\pi x_1), \tag{A5.7}$$

which can be expressed as in (A4.5) with ν running only over the set of vectors $I = \{\mu_j\}_{j=1}^4$, with

$$\mu_1 = (1, 1), \quad \mu_2 = (-1, -1), \quad \mu_3 = (1, 0), \quad \mu_4 = (-1, 0), \tag{A5.8}$$

corresponding to that where one has $\sigma_{1,\mu_1} = \sigma_{1,\mu_2} = 1$ and $\sigma_{1,\mu_3} = \sigma_{1,\mu_4} = \frac{1}{2}$.

The conditions $\mu \in I$ and $S_0^n\mu \in I$ can be satisfied only for (i) $\mu = \mu_3$ and $n = 1$ (so that $S_0^n\mu_3 = \mu_1$) and (ii) $\mu = \mu_4$ and $n = 1$ (so that $S_0^n\mu_4 = \mu_2$). Then, from (A5.3) and (A5.4), one has

$$\left.\frac{\partial^2\langle\sigma\rangle}{\partial\varepsilon^2}\right|_{\varepsilon=0} = \overline{\sigma_{1,\mu_3}}\sigma_{1,\mu_1} + \overline{\sigma_{1,\mu_4}}\sigma_{1,\mu_2} = 1, \tag{A5.9}$$

which shows that for ε small enough the volume contraction rate is positive, i.e. the system (A5.1) with perturbation (A5.5) is dissipative. The system (A5.1) is not reversible, but a reversible system can be easily constructed as explained in [**13**, §2]: if $(\mathbb{T}, S)$ is the system (A5.1), then we can define a new system $(\mathbb{T} \times \mathbb{T}, S')$, such that $S'(\mathbf{x}, \mathbf{y}) = (S\mathbf{x}, S^{-1}\mathbf{y})$,

† Note that, even if (A5.4) vanishes, this does not yield $\langle\sigma\rangle = 0$, but only that the second derivative of $\langle\sigma\rangle$ with respect to ε is vanishing in $\varepsilon = 0$.

with $(\mathbf{x}, \mathbf{y}) \in \mathbb{T} \times \mathbb{T}$, and the isometry $\mathcal{I}$ such that $\mathcal{I}^2 = 1\!\!1$ and $S^n\mathcal{I} = \mathcal{I}S^{-n}$ is given by $\mathcal{I}(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$. This means that a new reversible and (still) dissipative system is obtained, and the fluctuation theorem in [§2], follows.

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