

A Kac system interacting with two heat reservoirs: the shearing case

Federico Bonetto^{1,*} and Matthew Powell¹

¹ School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

* corresponding author: bonetto@math.gatech.edu

June 29, 2026

Abstract

We study a system formed by M particles moving in 3 dimensions and interacting with two heat reservoirs, each with $N \gg M$ particles. The system and the reservoirs interact via random collisions and thus evolve via a Kac-type master equation. The initial state of the reservoirs is given by two non-centered Maxwellian distributions; they have temperature T_+ and T_- and have average velocity $\vec{p}_+$ and $\vec{p}_-$, respectively. We prove that, for times shorter than $\sqrt{N}/M$, the interaction with the two reservoirs is well-approximated by the interaction with two shearing *dynamic* Maxwellian thermostats (i.e. heat reservoirs with $N = \infty$). As a byproduct of our analysis, we obtain a uniform in time approximation when $T_+ = T_-$ and $\vec{p}_+ = \vec{p}_-$.

Acknowledgement

We are indebted to Omar Hurtado, Michael Loss and Tobias Ried for many enlightening discussions.

1 Introduction, Model and Results

The Kac model describes particles interacting via random binary collisions meant to give a simplified picture of what happens in a real gas. The model was introduced in 1953 by Mark Kac[1]; the hope was to use the model to prove the existence of solutions for the Boltzmann equation. Although this has not been realized, the model has proven very useful in several different respects. In kinetic theory it helped reveal how systems approach equilibrium and the relation between the different norms one can use to measure such an approach, see [2, 3] for convergence in L^2 , [4] for convergence in L^1 and [5] for the GTW metric d_2 . It also helped clarify the conditions for the emergence of macroscopic behaviors like propagation of chaos and the validity of the Boltzmann-Kac equation, see e.g. [6, 7, 8]. Finally, the ideas behind the Kac evolution are partially present in the DSMC scheme for numerical simulations, see [9].

In a series of papers (see [10, 11, 12, 13]) we have tried to extend the reach of the Kac model by adding interactions both with reservoirs (still modeled via random collisions) and Maxwellian thermostats (seen as infinite reservoirs), and by looking at particles moving in three spacial dimensions, instead of one as in the original model. In particular, in [13] we looked at a three dimensional system with M

particles interacting with two large heat reservoirs with N particles each. The reservoirs are initially in equilibrium at temperatures T_+ and T_- , and we assumed that the centers of mass of both the system and the reservoirs are initially at rest. We proved that the system + 2 reservoirs, when $N \gg M$, that is the reservoirs are much larger than the system, is well approximated by a system of M particles interacting with 2 Maxwellian thermostats. This means that we can formally take the limit for $N \rightarrow \infty$ and replace the collisions of the particle in the system with particles in the reservoirs with collisions with *virtual* particles randomly selected from Maxwellian distributions with temperature T_+ or T_- . Contrary to the case of only one reservoir, see [11], our results in [13] were only valid for a time shorter than $\sqrt{N}/M$. Observe that, by taking the limit $N \rightarrow \infty$, we eliminate the influence of the system on the reservoirs. This leads to the notion of a thermostat with fixed temperature which was analyzed in [11]. On the other hand, for any fixed N , over a time of the order of N/M the temperatures of the reservoir and system evolve. That is, no matter how large (but finite) the reservoirs are, there will be temperature evolution. Thus the approximation with thermostats at fixed temperature can only be valid for a relatively short time.

In this paper, we continue this line of study by considering a generalization of the model considered in [13] in which the centers of mass of the two reservoirs initially have non-zero velocities $\vec{p}_+$ and $\vec{p}_-$, in addition to having different temperatures. If these velocities are different, they will create a shear in the system. On a macroscopic level, the energy stored in the center of mass motion of the reservoirs is converted into heat so that, in this case, the final temperature of the reservoirs is higher than the average of the initial temperatures. It is remarkable that one can write closed evolution equations for the momentum of the centers of mass and the temperatures of the system and reservoirs, see (9) and (10) below. As discussed in Subsection 2.1, two time scales emerge. On a short time scale of order 1 the system reaches a quasi stationary state while on a longer time scale of order N/M the temperatures and center of mass velocities of the two reservoirs tend to become equal and the all system + 2 reservoirs converges to equilibrium.

We can thus attempt to take the limit $N \rightarrow \infty$ in a more careful way. To create our effective evolution we assume that the particles in the system interact with 2 infinite reservoirs whose center of mass velocities and temperatures evolve in the same way as the corresponding quantities for the reservoirs with finite N . This give rise to the notion of *dynamic* thermostat as introduced in Subsection 1.3 below. More precisely, we compare the evolution of the system + 2 reservoir with that of a system interacting with 2 Maxwellian thermostat with center of mass velocities and temperatures, which evolves in time. We believe that the notion of dynamic thermostat is the major conceptual novelty of this paper and that thermostats of this type will be instrumental in extending our results beyond the time range obtained here.

Our main result is contained in Theorem 1 below. Similarly to [13], we obtain that the system + 2 reservoirs is well approximated by the system + 2 dynamic thermostats for times shorter than $\sqrt{N}/M$. On a technical level, as in our previous works, we use the GTW d_2 metric (see [14]) to measure the distance between the reservoir and thermostat evolution. Observe that the d_2 metric requires the the distributions being compared have the same first moment. However, as already noticed, the first moment of the distributions of the system and/or the reservoirs evolve in time so that if the first moment of a thermostat remained fixed the d_2 -distance between the reservoir evolution and a fixed thermostat

evolution would diverge immediately. This makes the introduction of the dynamic thermostat necessary also at a technical level, at least as far as the center of mass velocities are involved. To prove our main theorem we follow the strategy of [13] based on a Duhamel expansion and on the functional inequality introduced in [11, Lemma 3.3]. After this, we are left with estimating the evolution of the thermostated system. Due to the presence of the dynamic thermostats, this last step is considerably more involved than the corresponding estimates in [13]. In this sense, Subsection 3.2 is the novel technical achievement of this paper. It also makes clear that the main obstruction to a result uniform in time is the variation of the state of the reservoirs/thermostats over the long time scale.

We also observe that if we suppose that the two reservoirs have equal temperatures ($T_+ = T_-$) and initial velocities ($\vec{p}_+ = \vec{p}_-$), then the system is essentially equivalent to a system interacting with one shearing reservoir. In this case there is no long time scale and both the shear and the temperature of the reservoir change very little along the evolution. Thus we are again able to obtain estimates uniform in time, see Corollary 2. We note that the system involving only one thermostat can be analyzed using a suitable L^2 norm, see [10, 15]. Although the analysis is simpler, the L^2 norm, as discussed for example in [10], is not well suited for systems with many degrees of freedom. Moreover, it is unclear how to extend these L^2 results to the case of a system interacting with more than one reservoir, even for a short time.

In Subsections 1.1 and 1.2 we introduce the Kac evolution for the system and the reservoirs. These are included for the sake of completeness but can be skipped by the reader familiar with this family of models. In Subsection 1.3 we introduce the notion of dynamic thermostat and in Subsection 1.4 we state our main result. Section 2 contains a few comments on and possible extensions of our results. Section 3 contains the main components of the proof of Theorem 1. Finally, the appendices deal with a few technical statements used in the proof.

1.1 The Kac dynamics

The basic ingredient for the Kac model evolution is the operator which describes the effect of a collision between two particles. Suppose that the particles are spherical and that the unit vector associated with the relative position of their centers is ω . If $\vec{v}_1$ and $\vec{v}_2$, with $\vec{v}_i = (v_{i,1}, v_{i,2}, v_{i,3}) \in \mathbb{R}^3$, are the outgoing velocities immediately after the collision, then the incoming velocities before the collision were

$$\vec{v}_1^*(\omega) = \vec{v}_1 - \langle (\vec{v}_1 - \vec{v}_2), \omega \rangle \omega, \quad \vec{v}_2^*(\omega) = \vec{v}_2 - \langle (\vec{v}_2 - \vec{v}_1), \omega \rangle \omega, \quad (1)$$

where $\langle \vec{v}, \vec{w} \rangle$ denotes the usual inner product in $\mathbb{R}^3$. In the following, we will use $\langle \cdot, \cdot \rangle$ to denote the usual inner product on any $\mathbb{R}^K$. The precise space under consideration will be clear from the context. Suppose now that before the collision the velocities of the two particles were distributed according to the probability distribution $g(\vec{v}_1, \vec{v}_2)$ on $\mathbb{R}^6$. If the unit vector ω is chosen randomly and uniformly on the unit sphere $\mathbb{S}^2$ then the effect of the collision on g is given by

$$R[g](\vec{v}_1, \vec{v}_2) = \int_{\mathbb{S}^2} g(\vec{v}_1^*(\omega), \vec{v}_2^*(\omega)) d\omega. \quad (2)$$

With this collision in hand, we can now start to describe the evolution in the models we will consider.

We consider a system of M particles moving in $\mathbb{R}^3$ with velocities $\underline{v} = (\vec{v}_1, \dots, \vec{v}_M) \in \mathbb{R}^{3M}$ where $\vec{v}_i \in \mathbb{R}^3$. Since the collisions occur randomly and are independent of the positions of the particles, the initial state of the system is described by a probability distribution $f_0(\underline{v})$ on $\mathbb{R}^{3M}$. We also assume that the particles are identical, so that f_0 is invariant under particle permutations.

Contrary to our previous works, see e.g. [13, 11], we assume that the center of mass of our system initially moves with average velocity $\vec{p}_S$ given by

$$\vec{p}_S := \frac{1}{N} \sum_i \int \vec{v}_i f_0(\underline{v}) d\underline{v} = \int \vec{v}_1 f_0(\underline{v}) d\underline{v}.$$

Since the collision described in (1) preserves the total momentum of the two particles, this assumption will become relevant only after we introduce the reservoirs, see Subsection 1.2 below. For the sake of notation we will write

$$f(\underline{v}) = \tilde{f}(\vec{v}_1 - \vec{p}_S, \dots, \vec{v}_M - \vec{p}_S) = \tilde{f}(\underline{v} - \underline{p}_S) \quad (3)$$

where we set $\underline{p}_S = (\vec{p}_S, \dots, \vec{p}_S)$. Finally we assume that the initial temperature T_S of the system, or better the average kinetic energy per degree of freedom w.r.t. the center of mass, is finite, that is

$$T_S := \frac{1}{3} \int_{\mathbb{R}^{3M}} \|\vec{v}_1 - \vec{p}_S\|^2 f(\underline{v}) d\underline{v} < \infty$$

The effect of a collision between particle i and j on any distribution f is described by the operator R in (2) acting on $\vec{v}_i$ and $\vec{v}_j$. That is

$$R_{i,j}^S[f](\underline{v}) = \int_{\mathbb{S}^2} f(\underline{v}_{i,j}(\omega)) d\sigma(\omega)$$

where

$$\underline{v}_{i,j}(\omega) = (\vec{v}_1, \dots, \vec{v}_i^*(\omega), \dots, \vec{v}_j^*(\omega), \dots, \vec{v}_M),$$

with $\vec{v}_i^*(\omega)$ and $\vec{v}_j^*(\omega)$ given by (1).

We assume, as in Kac's original work [1], that the collisions take place according to a Poisson process of intensity λ_M and that when a collision takes place, the pair of colliding particles is chosen randomly and uniformly. Choosing $\lambda_M = M\lambda_S/2$ we get that the infinitesimal generator for the evolution of a system with distribution f is

$$\mathcal{L}_S[f] = \frac{\lambda_S}{M-1} \sum_{i < j} (R_{i,j}^S[f] - f). \quad (4)$$

With the above choice of λ_M the average number of collisions that a particle suffers in a unit time is equal to λ_S independently of M . λ_S^{-1} is called the *mean free flight* for the system.

1.2 Shearing Reservoirs

We consider the situation when a system with M particles interacts with two much larger reservoirs each containing N particles with $N \gg M$. We will assume that the two reservoirs are initially in canonical equilibrium at temperature T_+ and T_- and that their centers of mass move with average velocities $\vec{p}_+$ and $\vec{p}_-$. We call $\underline{u}_+ = (\vec{u}_{+,1}, \dots, \vec{u}_{+,N}) \in \mathbb{R}^{3N}$ the velocities of the particles in the $+$ reservoir and similarly $\underline{u}_- = (\vec{u}_{-,1}, \dots, \vec{u}_{-,N}) \in \mathbb{R}^{3N}$ for the particles in the $-$ reservoir. To shorten notation we will also use $\vec{u}_i = (\vec{u}_{+,i}, \vec{u}_{-,i}) \in \mathbb{R}^6$ and $\underline{u} = (\vec{u}_1, \dots, \vec{u}_N) \in \mathbb{R}^{6N}$ for the combined variables of both reservoirs.

The state of the combined system + 2 reservoirs is now given by a distribution $F(\underline{u}, \underline{v})$ on $\mathbb{R}^{3(M+2N)}$ and the initial state is of the form

$$F_0(\underline{v}, \underline{u}) = \Gamma_+^N(\underline{u}_+ - \underline{p}_+) f_0(\underline{v}) \Gamma_-^N(\underline{u}_- - \underline{p}_-) \quad (5)$$

where, for $\sigma \in \{+, -\}$, we set $\underline{p}_\sigma = (\vec{p}_\sigma, \dots, \vec{p}_\sigma)$ and

$$\Gamma_\sigma^N(\underline{u}_\sigma) = \Gamma_{T_\sigma}^N(\underline{u}_\sigma) = \prod_{i=1}^N \Gamma_{T_\sigma}(\vec{u}_{\sigma,i})$$

with

$$\Gamma_T(\vec{v}) = (2\pi T)^{-\frac{3}{2}} e^{-\frac{\|\vec{v}\|^2}{2T}}.$$

We can write the generator of the combined system + 2 reservoirs evolution as

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_{R_+} + \mathcal{L}_{R_-} + \mathcal{L}_{I_+} + \mathcal{L}_{I_-} \quad (6)$$

where $\mathcal{L}_S$ still describes the internal evolution of the system, see (4), while $\mathcal{L}_{R_\sigma}$ and $\mathcal{L}_{I_\sigma}$, with $\sigma \in \{+, -\}$, describe the internal evolution of the σ reservoir and the interaction of the σ reservoir with the system, respectively. This means that

$$\mathcal{L}_{R_\sigma}[F] = \frac{\lambda_R}{N-1} \sum_{i < j} (R_{i,j}^{R_\sigma}[F] - F)$$

with $R_{i,j}^{R_\sigma}$ describing the effect of a collision in the σ reservoir between particle i with velocity $\vec{u}_{\sigma,i}$ and particle j with velocity $\vec{u}_{\sigma,j}$ via the operator (2) acting on $\vec{u}_{\sigma,i}$ and $\vec{u}_{\sigma,j}$. Finally $\mathcal{L}_{I_\sigma}$ describes the interaction between the σ reservoir and the system. It is also modeled via a Kac-style collision and we set

$$\mathcal{L}_{I_\sigma}[F] = \frac{\mu}{N} \sum_{i=1}^N \sum_{j=1}^M (R_{i,j}^{I_\sigma}[F] - F) \quad (7)$$

where $R_{i,j}^{I_\sigma}$ describes the effect of a collision between particle i in the σ reservoir with velocity $\vec{u}_{\sigma,i}$ and particle j in the system with velocity $\vec{v}_j$. It is again given by the operator (2) acting on $\vec{u}_{\sigma,i}$ and $\vec{v}_j$. The factor μ/N in (7) is chosen so that a particle in the system suffers, on average, μ collisions per unit time with particles in the reservoir. At the same time, a particle in the reservoir on average suffers

$\mu M/N$ collision per unit time with particles in the system. Observe that there is no direct interaction between the two reservoirs.

The state of the system + 2 reservoirs at time t is thus given by

$$F_t = e^{\mathcal{L}t} F_0. \quad (8)$$

For every N and M , $\mathcal{L}$ is a bounded operator on $L^1(\mathbb{R}^{3M+2N})$ so that the evolution (8) is well defined for every t . Finally, for simplicity of notation, we choose units of time such that $\mu = 1$.

1.3 Dynamic Shearing Thermostats

In the spirit of [13], we want to obtain an effective approximation for the evolution of the system + 2 reservoirs considered in the previous subsection when N is very large. To do this we first need a better understanding of its macroscopic evolution. We can define the average velocities of the centers of mass at time t as

$$\begin{aligned} \vec{m}_S(t) &:= \frac{1}{M} \sum_{i=1}^N \int \vec{v}_i F_t(\underline{v}, \underline{u}) \, d\underline{v} d\underline{u} = \int \vec{v}_1 F_t(\underline{v}, \underline{u}) \, d\underline{v} d\underline{u} \\ \vec{m}_\sigma(t) &:= \frac{1}{N} \sum_{i=1}^N \int \vec{u}_{\sigma,i} F_t(\underline{v}, \underline{u}) \, d\underline{v} d\underline{u} = \int \vec{u}_{\sigma,1} F_t(\underline{v}, \underline{u}) \, d\underline{v} d\underline{u}. \end{aligned}$$

For simplicity sake, we will call $\vec{m}_S$ the *velocity of the system* and similarly for $\vec{m}_\sigma$. From the properties of the generator $\mathcal{L}$, see (6), it is not hard to derive explicit closed equations for the evolution of $\vec{m}_S$ and $\vec{m}_\sigma$, see [13] for more details. Indeed we get the Initial Value Problem

$$\frac{d}{dt} \begin{pmatrix} \vec{m}_+ \\ \vec{m}_S \\ \vec{m}_- \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -\frac{M}{N} & \frac{M}{N} & 0 \\ 1 & -2 & 1 \\ 0 & \frac{M}{N} & -\frac{M}{N} \end{pmatrix} \begin{pmatrix} \vec{m}_+ \\ \vec{m}_S \\ \vec{m}_- \end{pmatrix}, \quad \begin{pmatrix} \vec{m}_+(0) \\ \vec{m}_S(0) \\ \vec{m}_-(0) \end{pmatrix} = \begin{pmatrix} \vec{p}_+ \\ \vec{p}_S \\ \vec{p}_- \end{pmatrix} \quad (9)$$

To simplify the coming discussion we will chose a reference frame in which the combined center of mass of the system + 2 reservoirs is initially at rest, that is $N\vec{p}_+ + N\vec{p}_- + M\vec{p}_S = 0$. This condition is clearly preserved in time.

Similarly, the temperatures, or better the average kinetic energies per degree of freedom w.r.t. the centers of mass, at time t can be defined as

$$\begin{aligned} \tau_S(t) &:= \frac{1}{3} \int \|\vec{v}_1 - \vec{m}_S(t)\|^2 F_t(\underline{v}, \underline{u}) \, d\underline{v} d\underline{u} \\ \tau_\sigma(t) &:= \frac{1}{3} \int \|\vec{u}_{\sigma,1} - \vec{m}_\sigma(t)\|^2 F_t(\underline{v}, \underline{u}) \, d\underline{v} d\underline{u}. \end{aligned}$$

and they satisfy the IVP

$$\frac{d}{dt} \begin{pmatrix} \tau_+ \\ \tau_S \\ \tau_- \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -\frac{M}{N} & \frac{M}{N} & 0 \\ 1 & -2 & 1 \\ 0 & \frac{M}{N} & -\frac{M}{N} \end{pmatrix} \begin{pmatrix} \tau_+ \\ \tau_S \\ \tau_- \end{pmatrix} + \frac{1}{9} \begin{pmatrix} \frac{M}{N} \|\vec{m}_+ - \vec{m}_S\|^2 \\ \|\vec{m}_+ - \vec{m}_S\|^2 + \|\vec{m}_S - \vec{m}_-\|^2 \\ \frac{M}{N} \|\vec{m}_S - \vec{m}_-\|^2 \end{pmatrix}, \quad \begin{pmatrix} \tau_+(0) \\ \tau_S(0) \\ \tau_-(0) \end{pmatrix} = \begin{pmatrix} T_+ \\ T_S \\ T_- \end{pmatrix} \quad (10)$$

Let $\vec{m}_\sigma(t)$ and $\vec{m}_S(t)$ be the solutions of (9) and let $\underline{m}_S(t) = (\vec{m}_S(t), \dots, \vec{m}_S(t)) \in \mathbb{R}^{3M}$ and $\underline{m}_\sigma(t) = (\vec{m}_\sigma(t), \dots, \vec{m}_\sigma(t)) \in \mathbb{R}^{3N}$. Similarly let $\tau_S(t)$ and $\tau_\sigma(t)$ be the solution of (10). In Subsection 2.1 below we will give a more detailed discussion of these equations, their explicit solutions, and their physical meaning. We observe here that

$$\begin{aligned} \lim_{t \rightarrow \infty} \vec{m}_S(t) &= \lim_{t \rightarrow \infty} \vec{m}_\pm(t) = 0 \\ \lim_{t \rightarrow \infty} \tau_S(t) &= \lim_{t \rightarrow \infty} \tau_\pm(t) = \frac{N(T_+ + T_-) + MT_S}{N^*} + \frac{N\|\vec{m}_+ - \vec{m}_S\| + N\|\vec{m}_+ - \vec{m}_S\|}{3N^*} =: \tau_\infty. \end{aligned} \quad (11)$$

where $N^* = 2N + M$ is the total number of particles. This means that the system +2 reservoirs eventually reaches an equilibrium state where the the system and the 2 reservoirs are at rest and have the same temperature.

We are now ready to introduce our effective evolution. The following heuristic discussion will lead us to the evolution equation (16) for the system interacting with 2 *dynamic* thermostats. The validity of the approximations discussed below will be proved *a posteriori* in Theorem 1, at least for a not too long time

From (9) and (10), see also Subsection 2.1, we see that $\vec{m}_\sigma(t)$ and $\tau_\sigma(t)$ vary slowly in time with a rate of the order of M/N . On the other hand, the Kac collisions between particle in the same reservoir tend to keep the state of the reservoir close to a distribution that is rotational invariant in the direction orthogonal to the total momentum. Finally, since N is very large and the initial state of each reservoir is a product state, we expect that the Kac collision inside the reservoirs will tend to push the state of the reservoirs toward distributions which are chaotic, in the sense that they remain close to product states. See [1] for more details on chaotic distributions.

From the above discussion, we expect that

$$F_t(\underline{v}, \underline{u}) \simeq G_{+,t}^N(\underline{u}_+) f_t(\underline{v}) G_{-,t}^N(\underline{u}_-) \quad (12)$$

where for $\sigma \in \{+, -\}$ we set

$$G_{\sigma,t}^N(\underline{u}_\sigma) := \prod_{i=1}^N G_{\sigma,t}(\underline{u}_{\sigma,i}) \quad \text{with} \quad G_{\sigma,t}(\underline{u}_{\sigma,i}) := \Gamma_{\tau_\sigma(t)}(\vec{u}_{\sigma,i} - \vec{m}_\sigma(t)) \quad (13)$$

with $\vec{m}_S(t)$, $\vec{m}_\sigma(t)$ solution of (9) and $\tau_S(t)$ and $\tau_\sigma(t)$ solutions of (10). That is, we think that the reservoirs evolve slowly in time and remain always close to the canonical state defined by the solution of (9) and (10).

In light of this, we are led to conjecture that the evolution of the system can be effectively described via the interaction with two dynamic thermostats. That is, if the state of the system at time t is f_t , we consider the state

$$\tilde{F}_t(\underline{v}, \underline{u}) := G_{+,t}^N(\underline{u}_+) f_t(\underline{v}) G_{-,t}^N(\underline{u}_-)$$

formed by the system plus the two reservoirs in their effective state. We then compute the time variation of f_t by applying the generator $\mathcal{L}$, see (6), and take the marginal over the variables of the reservoirs.

That is, the evolution equation for f_t becomes:

$$\frac{d}{dt}f_t(\underline{v}) = \int \mathcal{L}[\tilde{F}_t](\underline{v}, \underline{u}) d\underline{u}. \quad (14)$$

This means that the system feels the influence of two reservoirs whose state at time t is prescribed a priori by (13) and thus is not influenced by the state of the system. To be more precise, we define the one-parameter family of operators

$$\Psi(t)[f] = \int \mathcal{L}[G_{+,t}^N f G_{-,t}^N](\underline{u}_+, \underline{v}, \underline{u}_-) d\underline{u}_+ d\underline{u}_-$$

and the two-parameter family $\Phi_{t,s}$, $t \geq s$, that satisfies

$$\frac{d}{dt}\Phi_{t,s} = \Psi(t)\Phi_{t,s}, \quad \Phi_{t,t} = \text{Id} \quad (15)$$

Thus the solution of (14) can be written as

$$f_t = \Phi_{t,0}f_0 \quad (16)$$

and hence the evolution of the system + 2 thermostats is given by

$$\tilde{F}_t(\underline{v}, \underline{u}) = G_{+,t}^N(\underline{u}_+) \Phi_{t,0}[f_0](\underline{v}) G_{-,t}^N(\underline{u}_-)$$

As for $\mathcal{L}$, it is not hard to see that $\Psi(t)$ is a bounded operator for every t so that $\Phi_{t,s}$ is well defined for every $s \leq t$. Moreover we can define

$$\vec{m}_S(t) := \int \vec{v}_1 \tilde{F}_t(\underline{v}, \underline{u}) d\underline{v} d\underline{u} \quad \text{and} \quad \tilde{\tau}_S(t) := \frac{1}{3} \int \|\vec{v}_1 - \vec{m}_S(t)\|^2 \tilde{F}_t(\underline{v}, \underline{u}) d\underline{v} d\underline{u} \quad (17)$$

and similar definitions for $\vec{m}_\sigma(t)$ and $\tilde{\tau}_\sigma(t)$. Notice that $\vec{m}_S$ and $\tilde{\tau}_S$ satisfy the same differential equation as $\vec{m}_S$ and τ_S in (9) and (10) with the same initial conditions while $\vec{m}_\sigma(t)$ and $\tilde{\tau}_\sigma(t)$ are clearly identical with $\vec{m}_\sigma(t)$ and $\tau_\sigma(t)$. It thus follow that

$$\vec{m}_S(t) = \vec{m}_S(t) \quad \text{and} \quad \tilde{\tau}_S(t) = \tau_S(t)$$

for every t , that is the macroscopic evolution of the system + 2 thermostats coincides with that of the system + 2 reservoirs. Observe finally that, due to the product nature of $G_{\sigma,t}^N$, $\Psi(t)$ does not depend on N and it can be written in a form very similar to the *static* thermostat studied in [10, 11, 13], see Subsection 2.2 for more details.

In the case in which $T_+ = T_- =: T$ and $\vec{p}_+ = \vec{p}_- =: \vec{p}$, we can imagine that the two reservoirs constitute a single reservoir with $2N$ particles at temperature T and velocity $\vec{p}$. Moreover, if we fix the initial velocity of the system + reservoir to be 0 we get $\vec{p} = -\frac{M}{2N}\vec{p}_S$. Thus, if $N \gg M$, we can imagine that the reservoir is effectively at rest and interacts with a system that is initially in motion with velocity $\vec{p}_S$. It is easy to see from (9) and (10) that in this situation the temperature and the velocity of the reservoir vary very little along the evolution. For these reasons, in this particular case, we will be able to obtain estimates that are uniform in time, see Corollary 2 below.

1.4 Main Result: comparing F_t and $\tilde{F}_t$

We want to compare the system + 2 reservoir evolution F_t given by (8) with the system + two thermostat evolution $\tilde{F}_t$ given by (12) with f_t defined by (16). As in [13], we will compare F_t and $\tilde{F}_t$ using the GTW d_2 distance, see [14], that we briefly introduce and discuss below.

For $f : \mathbb{R}^K \mapsto \mathbb{R}$ we write its Fourier transform as

$$\hat{f}(\underline{\xi}) = \int_{\mathbb{R}^K} e^{2\pi i \langle \underline{\xi}, \underline{v} \rangle} f(\underline{v}) d\underline{v}.$$

Given two distributions $f, g : \mathbb{R}^K \mapsto \mathbb{R}$ with

$$\int_{\mathbb{R}^K} f(\underline{v}) d\underline{v} = \int_{\mathbb{R}^K} g(\underline{v}) d\underline{v} = 1 \quad \text{and} \quad \int_{\mathbb{R}^K} \underline{v} f(\underline{v}) d\underline{v} = \int_{\mathbb{R}^K} \underline{v} g(\underline{v}) d\underline{v} \quad (18)$$

we define the d_2 distance between f and g as

$$d_2(f, g) = \sup_{\underline{\xi} \neq 0} \frac{|\hat{f}(\underline{\xi}) - \hat{g}(\underline{\xi})|}{\|\underline{\xi}\|^2}. \quad (19)$$

Notice that the two conditions in (18) are necessary for the supremum in (19) to be finite. On the other hand if we further assume that

$$\int_{\mathbb{R}^K} \|\underline{v}\|^2 f(\underline{v}) d\underline{v}, \int_{\mathbb{R}^K} \|\underline{v}\|^2 g(\underline{v}) d\underline{v} < \infty$$

then we get that $d_2(f, g) < \infty$. Finally we observe that the d_2 distance is unchanged if we change the average velocity. That is, for any $\underline{m} \in \mathbb{R}^K$ we have

$$d_2(f(\cdot - \underline{m}), g(\cdot - \underline{m})) = d_2(f, g). \quad (20)$$

Thus the choice of a particular reference frame, as discussed after (9), has no effect on our results.

Like our result in [13], in the case of two thermostat we are unable to obtain a uniform estimate in t due to the fact that the states of the reservoirs change in time. Notwithstanding the temperatures and velocities of our dynamic thermostats exactly follow the temperatures and velocities of the reservoirs, this is apparently not enough. See Subsection 2.1 for a more detailed discussion of what we think is needed to obtain a uniform estimate.

We formulate our results in terms of the distance $d_2(\tilde{f}_0, \Gamma_{\tau_\infty}^M)$ between the recentered initial distribution $\tilde{f}_0$, see (3), and the Maxwellian at equilibrium temperature τ_∞ , together with the two quantities that characterize how far from equilibrium the reservoirs are at time $t = 0$:

$$\Delta_p = \|\vec{p}_+ - \vec{p}_S\|^2 + \|\vec{p}_- - \vec{p}_S\|^2 \quad \text{and} \quad \Delta_T = |T_+ - T_S| + |T_- - T_S|. \quad (21)$$

We are now ready for our main Theorem.

Theorem 1. Let f_0 be a distribution on $\mathbb{R}^{3M}$, $T_+, T_- \in \mathbb{R}$ and $\vec{p}_+, \vec{p}_- \in \mathbb{R}^3$ and consider the initial state F_0 given by (5) and its evolution F_t given by (8). Consider also the evolution of the initial state F_0 given by $\tilde{F}_t = G_{+,t}^N \Phi_{t,0}[f_0] G_{-,t}^N$ with $\Phi_{t,0}$ defined in (15). Then we have

$$d_2(F_t, \tilde{F}_t) \leq C(f_0) \frac{M}{\sqrt{N}} \left(d_2(f_0, \Gamma_{\tau_\infty}^M)^{\frac{1}{6}} \left(1 - e^{-\frac{t}{18}} \right) + (\Delta_p^2 + \Delta_T)^{\frac{1}{6}} t \right) \quad (22)$$

for all $t \geq 0$ and a suitable constant $C(f_0)$.

Remark 1. The constant $C(f_0)$ is (22) does not depend on N or M but it still depends on f_0 through its moments of low degree, including $\vec{p}_S$ and T_S , together with T_σ and $\vec{p}_\sigma$, see Lemma 1 and Subsection 3.2 below for more details. We chose to explicitly write only the dependence on quantities that are relevant for the non equilibrium character of our model.

In the case that the two thermostats have the same initial temperature and the same velocity we can obtain a better result.

Corollary 2. In the hypotheses of Theorem 1, if $T_+ = T_- =: T$ and $\vec{p}_+ = \vec{p}_-$ we get

$$d_2(F_t, \tilde{F}_t) \leq C \frac{M}{\sqrt{N}} \left(d_2(f_0, \Gamma_T^M)^{\frac{1}{6}} + (\Delta_p^2 + \Delta_T)^{\frac{1}{6}} \right) \left(1 - e^{-\frac{t}{18}} \right)$$

for all $t > 0$.

Observe that, even if the two reservoirs start with $T_+ = T_-$ but $\vec{p}_+ \neq \vec{p}_-$, the temperature of the reservoirs will change with time, see (9) and discussion in Subsection 2.1. Thus both the conditions on the temperatures and velocities are necessary for the validity of Corollary 2.

2 Discussion and Outlook

We collect here few comments on the meaning of our results and suggestions for possible extensions.

2.1 Hydrodynamics

It is not hard to see that the solution to (9) can be written as:

$$\begin{pmatrix} \vec{m}_+(t) \\ \vec{m}_S(t) \\ \vec{m}_-(t) \end{pmatrix} = \frac{\vec{p}_+ - \vec{p}_-}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-\frac{M}{3N}t} + \frac{\vec{p}_+ - 2\vec{p}_S + \vec{p}_-}{2N^*} \begin{pmatrix} M \\ -2N \\ M \end{pmatrix} e^{-\frac{1}{3}(2+\frac{M}{N})t} \quad (23)$$

where we have used our assumption that the total center of mass is initially at rest. Thus, on a time scale of order 1, $\vec{m}_S(t)$ goes to 0 while $\vec{m}_+(t)$ and $\vec{m}_-(t)$ become equal and opposite. On a longer time scale of order of N/M , $\vec{m}_\sigma(t)$ also tends to zero.

Inspection of equations (9) and (10) shows that the loss of kinetic energy by the center of mass motion of the reservoirs and system is compensated by the positive contribution to the temperature of the non

homogeneous terms in (10). Indeed, we can see from (23) that $\vec{m}_S(t)$ approaches its asymptotic value exponentially fast with a rate of order 1. The kinetic energy-temperature correspondence discussed above (or, indeed, direct inspection of (10)) leads us to see that $\tau_S(t)$ also approaches its asymptotic value exponentially fast with a rate of order 1. Thus, to understand the large time behavior of the solution of (10), we consider the special case in which $\vec{p}_S = 0$ so that $\vec{p}_+ = -\vec{p}_-$ while $T_S = (T_+ + T_-)/2$. In this situation we can write the solution of (10) as

$$\begin{aligned}\tau_+(t) &= T_S + \frac{2N}{3N^*} \|p_+\|^2 + \frac{T_+ - T_-}{2} e^{-\frac{M}{3N}t} + \frac{M}{3N^*} \|\vec{p}_+\|^2 e^{-\frac{1}{3}(2+\frac{M}{N})t} - \frac{\|p_+\|^2}{3} e^{-\frac{2M}{3N}t} \\ \tau_S(t) &= T_S + \frac{2N}{3N^*} \|p_+\|^2 - \frac{2N}{3N^*} \|p_+\|^2 e^{-\frac{1}{3}(2+\frac{M}{N})t} \\ \tau_-(t) &= T_S + \frac{2N}{3N^*} \|p_+\|^2 - \frac{T_+ - T_-}{2} e^{-\frac{M}{3N}t} + \frac{M}{3N^*} \|\vec{p}_+\|^2 e^{-\frac{1}{3}(2+\frac{M}{N})t} - \frac{\|p_+\|^2}{3} e^{-\frac{2M}{3N}t}\end{aligned}$$

where we clearly see that the final temperature of the system and reservoirs is greater than T_S due to the transformation of the kinetic energy of the centers of mass of the reservoirs into heat thanks to the *viscosity* terms in (10). Observe that if $T_+ = T_-$, after an initial transient, the temperature of the system is above the temperatures of the two reservoirs. That is, the heat produced by the friction between the two reservoirs due to their opposite motion heats up the system and then is diffused back to the reservoirs until equilibrium is reached.

2.2 Static vs dynamic thermostats

To make a connection with the definition of the thermostat in [11, 13], we observe that for $g : \mathbb{R}^3 \mapsto \mathbb{R}$ we can define

$$B_\sigma(t)[g](\vec{v}) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Gamma_{\tau_\sigma(t)}(\vec{u}_\sigma^* - \vec{m}_\sigma(t)) g(\vec{v}^*) d\omega d\vec{u}_\sigma = \int_{\mathbb{R}^3} R[\Gamma_{\tau_\sigma(t)}(\cdot - \vec{m}_\sigma(t))g](\vec{u}_\sigma, \vec{v}) d\vec{u}_\sigma \quad (24)$$

For $f : \mathbb{R}^{3M} \mapsto \mathbb{R}$, we can call $B_{k,\sigma(t)}[\tilde{f}]$ the operator (24) acting on $\vec{v}_k$, which is precisely the form of the thermostat defined in [11, 13]. Following [16], we can combine the two thermostats and define

$$B(t)[f](\vec{u}) := \int_{\mathbb{R}^3} R[\mathcal{M}_t f](\vec{u}, \vec{v}) d\vec{v} \quad (25)$$

where

$$\mathcal{M}_t(\vec{u}) := \frac{1}{2} (\Gamma_{\tau_+(t)}(\vec{u} - \vec{m}_+(t)) + \Gamma_{\tau_-(t)}(\vec{u} - \vec{m}_-(t)))$$

and obtain

$$\Psi(t)[f] = \mathcal{L}_S + 2 \sum_{k=1}^M (B_k(t)[f] - f) := \mathcal{L}_S[f] + \mathcal{L}_{B(t)}[f]. \quad (26)$$

where, as usual, $B_k(t)$ is the operator $B(t)$ acting on the k -th variable $\vec{v}_k$. We observe that (26) shows that $\Psi(t)$ does not depend on N . Moreover we only need $\mathcal{M}(t)$ to be a probability distribution for $\mathcal{L}_{B(t)}[f]$ to define a generator on the space of distributions.

In [13] we studied the system + 2 reservoirs described in Subsection 1.2 with the difference that we assumed $\vec{p}_\sigma = \vec{p}_S = 0$ and we compared its evolution with that of a system + 2 *static* thermostats, that is two thermostats at rest and with temperature fixed at its initial value. We observe that in the present paper we could not have used static thermostat. Indeed for the d_2 distance between F_t and $\tilde{F}_t$ to be finite we need $\vec{m}_S(t) = \vec{m}_S(t)$ and $\vec{m}_\sigma(t) = \vec{m}_\sigma(t)$, see (18) and (17).

On the other hand, we could have used dynamic thermostats with evolving velocity but fixed temperatures equal to the initial temperature of the reservoir. This would have lead to a result essentially identical to Theorem 1 with minor changes in the proof, see in particular (27) and (28). As already discussed in [13], we cannot hope to obtain a uniform estimate for the distance between F_t and $\tilde{F}_t$ without having a dynamic thermostat where the temperatures also evolve in time. On the other hand, the present paper shows that this is not enough in itself. Indeed, we can see that the unbounded term in Theorem 1 arises from the presence of a long time scale in the comparison between the evolution of the reservoirs and the evolution of the thermostats, see (39) and (40). To obtain uniform in time estimates, we would need to improve (33) to an order of $1/N^{1+\epsilon}$, for some $\epsilon > 0$, where it now reads $1/\sqrt{N}$. This is, actually, rather natural. Indeed, the thermostat we consider here is effectively a zeroth order in $1/N$ approximation of the Kac evolution, even with the moving thermostats. Hence, from an analytic point of view, using dynamic thermostats is not enough. In order to improve the estimates, we need to add to the dynamic thermostat a first-order approximation of the Kac evolution, which would probably arise from an analysis of the associated Boltzmann-Kac equation.

Observe finally that for any fixed t , the equation $\Psi(t)[\phi_t] = 0$ has a unique normalized solution. This is the non equilibrium steady state the for system + 2 thermostats at fixed velocities $\vec{m}_\sigma(t)$ and temperatures $\tau_\sigma(t)$. Since, for N large, $\vec{m}_\sigma(t)$ and $\tau_\sigma(t)$ evolve very slowly, it is natural to think that $f_t \simeq \phi_t$. That is, on a time scale much longer than N/M , the reservoirs evolve along the canonical states determined by the solution of (9) and (10) while the sytem evolve along a slowly varying sequence of non equilibrium steady states ϕ_t . This is the effective picture that we hope to establish in our forthcoming works.

3 Proof of the Theorem 1

We will begin with the same general set-up as in [13] and adjust the argument where necessary to account for the evolving thermostats.

3.1 Step 1: Duhamel expansion

Our goal here is to reduce the computation of $d_2(F_t, \tilde{F}_t)$ to a point where we may appeal to the analysis we performed in [13]. As in [13], we begin with a Duhamel expansion that yields

$$\begin{aligned} F_t - \tilde{F}_t &= \int_0^t e^{\mathcal{L}(t-s)} \left[\mathcal{L}[\tilde{F}_s] - \frac{d}{ds} \tilde{F}_s \right] ds \\ &= \int_0^t e^{\mathcal{L}(t-s)} \left[\mathcal{L}[\tilde{F}_s] - \Psi(s)[f_s] G_{+,s}^N G_{-,s}^N - f_s \frac{d}{ds} (G_{+,s}^N G_{-,s}^N) \right] ds \end{aligned}$$

Next, we pass the above into Fourier space. Let $\underline{\xi} = (\vec{\xi}_1, \dots, \vec{\xi}_N)$ be the Fourier variables associated with $\underline{v}$ while $\underline{\eta}_\sigma = (\vec{\eta}_{\sigma,1}, \dots, \vec{\eta}_{\sigma,1})$ are the Fourier variables associated with $\underline{u}_\sigma$. We will also write $\underline{\eta} = (\vec{\eta}_1, \dots, \vec{\eta}_N)$ with $\vec{\eta}_i = (\vec{\eta}_{+,1}, \vec{\eta}_{-,1})$ for the combined Fourier variables of the two reservoirs. We obtain

$$\hat{F}_t - \hat{\tilde{F}}_t = \int_0^t e^{\mathcal{L}(t-s)} \left[\mathcal{L}[\hat{F}_s] - \hat{\Psi}(s)[\hat{f}_s] \hat{G}_{+,s}^N \hat{G}_{-,s}^N - \sum_{\sigma \in \{+, -\}} \left(2\pi i \langle \underline{\eta}_\sigma, \dot{\underline{m}}_\sigma(s) \rangle + \frac{1}{2} \|\underline{\eta}_\sigma\|^2 \dot{\tau}_\sigma(s) \right) \hat{\tilde{F}}_s \right] ds$$

where we have used that $\hat{\mathcal{L}} = \mathcal{L}$. Moreover, for $\sigma \in \{+, -\}$,

$$\hat{G}_{\sigma,s}^N(\underline{\eta}_\sigma) = \prod_{i=1}^N \hat{G}_{\sigma,s}(\vec{\eta}_{\sigma,i}) \quad \text{with} \quad \hat{G}_{\sigma,s}(\vec{\eta}) = e^{-2\pi i \langle \vec{\eta}, \vec{m}_\sigma(t) \rangle} e^{-\frac{\tau_\sigma(s)}{2} \|\vec{\eta}\|^2}$$

is the Fourier transform of $G_{\sigma,s}^N$ while $\hat{\Psi}(s)$ is the Fourier transform of $\Psi(s)$ and it is given by

$$\hat{\Psi}(s)[\hat{f}](\underline{\xi}) = \mathcal{L}[\hat{G}_{s,-}^N \hat{f} \hat{G}_{+,s}^N](\underline{\xi}, 0).$$

Thus we obtain

$$d_2(F_t, \tilde{F}_t) \leq \sup_{\underline{\xi}, \underline{\eta} \neq 0} \frac{1}{\|\underline{\xi}\|^2 + \|\underline{\eta}\|^2} \int_0^t ds \left| \mathcal{L}[\hat{F}_s](\underline{\xi}, \underline{\eta}) - \hat{\Psi}(s)[\hat{f}_s](\underline{\xi}) \hat{G}_{+,s}^N(\underline{\eta}_+) \hat{G}_{-,s}^N(\underline{\eta}_-) - \sum_{\sigma \in \{+, -\}} \left(2\pi i \langle \underline{\eta}_\sigma, \dot{\underline{m}}_\sigma(s) \rangle + \frac{1}{2} \|\underline{\eta}_\sigma\|^2 \dot{\tau}_\sigma(s) \right) \hat{\tilde{F}}_s(\underline{\xi}, \underline{\eta}) \right|. \quad (27)$$

We first observe that

$$\sup_{\underline{\xi}, \underline{\eta} \neq 0} \frac{\sum_{\sigma \in \{+, -\}} \|\underline{\eta}_\sigma\|^2 |\dot{\tau}_\sigma(s)| |\hat{\tilde{F}}_s(\underline{\eta}_-, \underline{\xi}, \underline{\eta}_+)|}{\|\underline{\xi}\|^2 + \|\underline{\eta}\|^2} \leq \frac{M}{3N} \sum_{\sigma \in \{+, -\}} \left(|\tau_\sigma(s) - \tau_S(s)| + \frac{1}{3} \|\vec{m}_\sigma(s) - \vec{m}_S(s)\|^2 \right) \quad (28)$$

where we have used (10). To bound the remaining terms in the r.h.s of (27), mirroring our approach from [13], we can expand $\mathcal{L}$ into a suitable sum involving the collision operators $R_{i,j}^{I_\sigma}$. Since each collision only sees two of the particles at a time, it is convenient to ‘factor’ the remaining particles out of the collision to simplify the expression, so we begin by setting $\underline{\eta}_\sigma^k = (\vec{\eta}_{\sigma,1}, \dots, \vec{\eta}_{\sigma,k-1}, \vec{\eta}_{\sigma,k+1}, \dots, \vec{\eta}_{\sigma,N})$ and $\underline{\eta}^k = (\vec{\eta}_1, \dots, \vec{\eta}_{k-1}, \vec{\eta}_{k+1}, \dots, \vec{\eta}_N)$. Clearly, $\underline{\eta}^k$ is simply $\underline{\eta}$ with the k th component, $\vec{\eta}_k$, removed. For convenience of notation, we also define the operator H_s :

$$H_s[\hat{f}](\underline{\xi}, \vec{\eta}_k) = \sum_{j=1}^M \sum_{\sigma \in \{\pm\}} \left(R_{k,j}^{I_\sigma} [\hat{f} \hat{G}_{\sigma,s}^1](\underline{\xi}, \vec{\eta}_{\sigma,k}) - R_{k,j}^{I_\sigma} [\hat{f} \hat{G}_{\sigma,s}^1](\underline{\xi}, 0) G_{\sigma,s}^1(\vec{\eta}_{\sigma,k}) \right) G_{\sigma',s}^1(\vec{\eta}_{\sigma',k}) - 2\pi i N \hat{G}_{+,s}^1(\vec{\eta}_{+,k}) \hat{G}_{-,s}^1(\vec{\eta}_{-,k}) \sum_{\sigma \in \{+, -\}} \langle \vec{\eta}_{\sigma,k}, \dot{\vec{m}}_\sigma(s) \rangle \hat{f}(\underline{\xi}) \quad (29)$$

where $\sigma' = -\sigma$. With this, we can indeed ‘factor’ the integrand above and obtain

$$\begin{aligned} \mathcal{L}[\widehat{F}_s](\underline{\xi}, \underline{\eta}) - \widehat{\Psi}(s)[\hat{f}_s](\underline{\xi}) \widehat{G}_s^+(\underline{\eta}_+) \widehat{G}_s^-(\underline{\eta}_-) - 2\pi i \sum_{\sigma \in \{\pm\}} \langle \underline{\eta}_\sigma, \underline{m}_\sigma(s) \rangle \widehat{F}_s(\underline{\xi}, \underline{\eta}) = \\ \frac{1}{N} \sum_{k=1}^N H_s[\hat{f}_s](\vec{\xi}_j, \vec{\eta}_k) \widehat{G}_{+,s}^{N-1}(\underline{\eta}_+^k) \widehat{G}_{-,s}^{N-1}(\underline{\eta}_-^k) \end{aligned}$$

Remark 2. Note that the first line in (29) is, at first sight, of order M while the second line apparently behaves differently (and in a problematic way) as a function of N and M . Closer inspection, however, reveals this is not the case since, from (9), we see that $\frac{d}{ds} \vec{m}_\sigma(s) = \frac{M}{3N} (\vec{m}_S(s) - \vec{m}_\sigma(s))$ and thus the second line in (29) is also of order M .

We can now define for any $K \geq 1$

$$\mathcal{D}_K(H, g, \underline{\xi}) := \sup_{\underline{\eta} \neq 0} \frac{\left| \sum_{i=1}^K H(\underline{\xi}, \vec{\eta}_i) \prod_{j \neq i}^K g(\vec{\eta}_j) \right|}{\|\underline{\xi}\|^2 + \|\underline{\eta}\|^2}. \quad (30)$$

so that setting $\widehat{G}_s^2(\vec{\eta}_k) = \widehat{G}_{+,s}(\vec{\eta}_{+,k}) \widehat{G}_{-,s}(\vec{\eta}_{-,k})$ we can finally write

$$\begin{aligned} \sup_{\underline{\xi}, \underline{\eta} \neq 0} \frac{1}{\|\underline{\xi}\|^2 + \|\underline{\eta}\|^2} \left| \mathcal{L}[\widehat{F}_s](\underline{\eta}_-, \underline{\xi}, \underline{\eta}_+) - \widehat{\Psi}(s)[\hat{f}_s](\underline{\xi}) \widehat{G}_{+,s}^N(\underline{\eta}_+) \widehat{G}_{-,s}^N(\underline{\eta}_-) - \right. \\ \left. 2\pi i \sum_{\sigma} \langle \underline{\eta}_\sigma, \underline{m}_\sigma(s) \rangle \widehat{F}_s(\underline{\eta}_-, \underline{\xi}, \underline{\eta}_+) \right| \leq \frac{1}{N} \mathcal{D}_N(H_s[\hat{f}_s], \widehat{G}_s^2, \underline{\xi}). \end{aligned}$$

A straightforward computation (and the fact that $\vec{m}_\sigma(s)$ solves (9)) verifies that

$$H_s[\hat{f}_s](\underline{\xi}, 0) = 0 \quad (31)$$

$$\nabla_{\vec{\eta}} H_s[\hat{f}_s](0, 0) = \vec{0}. \quad (32)$$

Moreover, we can define

$$\|H\|_{3,1} := \sup_{\|\vec{\alpha}_\eta\|_1 \leq 3, \|\vec{\alpha}_\xi\| \leq 1} \left\| \partial_{\vec{\eta}}^{\vec{\alpha}_\eta} \partial_{\vec{\xi}}^{\vec{\alpha}_\xi} H \right\|_\infty.$$

In [13], we estimated $\|H_s(\hat{f}_s)\|_{3,1}$ uniformly in time in the special case where $p_+ = p_S = p_- = 0$. In Appendix A, we will present an improved and streamlined analysis of $\|H_s(\hat{f}_s)\|_{3,1}$. Indeed, since the derivative of $\hat{f}_s$ are linked to the moments of f_s , by analyzing the latter we will show that, for a suitable constant C_2 ,

$$\sup_s \|H_s[\hat{f}_s]\|_{3,1} \leq C_2 M E_4(f_0)$$

where

$$E_K(f_0) := \sup_{\underline{\alpha} \in \mathbb{N}^{3M}, \|\underline{\alpha}\|_1 \leq K} \int \underline{v}^\alpha f(\underline{v}) f \underline{v}$$

and $\underline{\alpha} = (\vec{\alpha}_1, \dots, \vec{\alpha}_M)$, while

$$\underline{v}^\alpha = \prod_{i=1}^M \vec{v}_i^{\vec{\alpha}_i} \quad \text{with} \quad \vec{v}^\vec{\alpha} = \prod_{i=1}^3 v_i^{\alpha_i}$$

and

$$\|\underline{\alpha}\|_1 = \sum_{i=1}^3 \|\vec{\alpha}_i\|_1 \quad \text{with} \quad \|\vec{\alpha}\|_1 = \sum_{i=1}^3 \alpha_i.$$

Finally, a simple extension of our results in [13] gives rise to the following key technical Lemma:

Lemma 1. *Let $H(\underline{\xi}, \vec{\eta}) = H : \mathbb{R}^p \times \mathbb{R}^6 \rightarrow \mathbb{C}$ be such that*

$$\nabla_{\vec{\eta}} H(0, 0) = 0$$

and, for every $\underline{\xi} \in \mathbb{R}^p$,

$$H(\underline{\xi}, 0) = 0.$$

Assume moreover that

$$\|H\|_{3,1} < \infty.$$

Let also $g : \mathbb{R}^6 \mapsto \mathbb{C}$ be such that, for some T ,

$$|g(\vec{\eta})| \leq \frac{1}{1 + T\|\vec{\eta}^2\|}.$$

Then for $\mathcal{D}_N(H, g, \underline{\xi})$ defined in (30) we have

$$\mathcal{D}_N(H, g, \underline{\xi}) \leq C_1(T) \sqrt{N} \|H\|_{3,1}^{\frac{5}{6}} \mathcal{D}_1(H, g, \underline{\xi})^{\frac{1}{6}}.$$

for a suitable constant $C_1(T)$.

Remark 3. In [13, Lemma 3.3] we proved the above statement for $H(\vec{\xi}, \vec{\eta}) = H : \mathbb{R}^p \times \mathbb{R}^3 \rightarrow \mathbb{C}$. It is not hard to check that the proof work essentially without modifications for the situation of Lemma 1.

We thus obtain

$$d_2(F_t, \tilde{F}_t) \leq \frac{1}{\sqrt{N}} \int_0^t C \|H_s[\hat{f}_s]\|_{3,1}^{\frac{5}{6}} \sup_{\xi \neq 0} \left(\mathcal{D}_1(H_s[\hat{f}_s], \widehat{G}_s^2, \underline{\xi}) \right)^{\frac{1}{6}} ds \quad (33)$$

It remains for us to understand $\mathcal{D}_1(H_s[\hat{f}_s], \widehat{G}_{\pm,s}^2, \underline{\xi})$ and $\|H_s[\hat{f}_s]\|_{3,1}$. We will analyze the former in the next Subsection and, as we discussed above, the latter in Appendix A.

3.2 Step 2: estimating $\mathcal{D}_1(H_s[\hat{f}_s], \hat{G}_{\pm, s}^2, \underline{\xi})$

Observing that $\mathcal{D}_1(H_s[\hat{f}_s], \hat{G}_{\pm, s}^2, \underline{\xi}) = \sup_{\underline{\xi}, \vec{\eta}_1} |H_s[\hat{f}_s](\underline{\xi}, \vec{\eta}_1)| / (\|\underline{\xi}\|^2 + \|\vec{\eta}_1\|^2)$, it remains to understand

$$D_1 := \sup_{\underline{\xi}, \vec{\eta}_1} \frac{|H_s[\hat{f}_s](\underline{\xi}, \vec{\eta}_1)|}{\|\underline{\xi}\|^2 + \|\vec{\eta}_1\|^2}.$$

We reached a similar point in [13]. In that work, we were able to directly relate D_1 to the d_2 -distance between the initial state and the steady-state. That is not possible here, since the d_2 -distance between the initial state and the steady state is infinite when $\vec{m}_S \neq 0$, which is an important part of this paper. Moreover, such an analysis from [13] would obfuscate the various timescales present in the current evolution: the equilibration of the system, the evolution of the reservoir velocities, and the evolution of the reservoir temperatures. We will instead break-up $H_s[\hat{f}_s]$ into pieces which capture these different timescales.

We will do this in three steps. Observe first that, from Proposition 3 below, it follows that

$$\lim_{t \rightarrow \infty} f_t(\underline{v}) = \Gamma_{\tau_\infty}^M(\underline{v})$$

where τ_∞ is defined in (11). We thus define

$$\Gamma_{\infty, s}^M(\underline{v}) = \Gamma_{\tau_\infty}^M(\underline{v} - \underline{m}_S(s)).$$

That is, $\Gamma_{\infty, s}^M$ is the thermostated equilibrium state re-centered around $\underline{m}_S(s)$. Also let

$$h_s = \Phi_{s,0} \Gamma_{\infty,0}^M(\underline{v}).$$

That is, h_s starts at the same initial state as $\Gamma_{\infty, s}^M$ but evolves according to the thermostated evolution.

Since $H_s[f]$ is linear we can write

$$\begin{aligned} D_1 &\leq \sup_{\underline{\xi}, \vec{\eta}_1} \frac{|H_s[\hat{f}_s - \hat{h}_s](\underline{\xi}, \vec{\eta}_1)|}{\|\underline{\xi}\|^2 + \|\vec{\eta}_1\|^2} + \sup_{\underline{\xi}, \vec{\eta}_1} \frac{|H_s[\hat{h}_s - \hat{\Gamma}_{\infty, s}^M](\underline{\xi}, \vec{\eta}_1)|}{\|\underline{\xi}\|^2 + \|\vec{\eta}_1\|^2} + \sup_{\underline{\xi}, \vec{\eta}_1} \frac{|H_s[\hat{\Gamma}_{\infty, s}^M](\underline{\xi}, \vec{\eta}_1)|}{\|\underline{\xi}\|^2 + \|\vec{\eta}_1\|^2} = \\ &:= (I) + (II) + (III). \end{aligned}$$

Loosely speaking, we will show that (I) is bounded because $\Phi_{t,s}$ is a contraction; we will show that (II) is bounded via a second Duhamel expansion; and we will show that (III) is bounded due to the explicit form of H_s and $\Gamma_{\infty, s}^M$. This estimates will be based on several propositions.

Our estimate for (I), (II) and (III) will be in term of the shears $\delta_\sigma(s) := \vec{m}_\sigma(s) - \vec{m}_S(s)$ and temperatures gradients $\epsilon_\sigma(s) := \tau_\sigma(s) - \tau_S(s)$. Our first proposition relates these quantities with their initial values.

Proposition 1. *If $\vec{m}_\sigma$ and $\vec{m}_S$ solve the IVP (9) then we have*

$$\|\vec{\delta}_+(s)\|^2 + \|\vec{\delta}_-(s)\|^2 \leq e^{-\frac{2Ms}{3N}} \Delta_p. \quad (34)$$

Similarly if τ_σ and τ_S solve the IVP (9) then we have

$$|\epsilon_+(s)| + |\epsilon_-(s)| \leq e^{-\frac{2Ms}{3N}} \left(2\Delta_T + \frac{1}{3}\Delta_p \right). \quad (35)$$

Δ_p and Δ_T are defined in (21). Moreover we have

$$|\tau_\sigma(s) - \tau_\infty| \leq \frac{1}{2}(|\epsilon_+(s)| + |\epsilon_-(s)| + \|\vec{\delta}_+(s)\|^2 + \|\vec{\delta}_-(s)\|^2). \quad (36)$$

Proof. The simple proof of these statements is reported in Appendix B. $\square$

It will be useful for us to be able to relate $d_2(H_s[g], 0)$ to $d_2(g, 0)$ where appropriate, so we consider that first.

Remark 4. In this Subsection and in the Appendices we will use the letter C to indicate a generic constant independent of M and N . It is not supposed to have a fixed value even when it appears in the same formula multiple times.

Proposition 2. Let $g : \mathbb{R}^{3M} \mapsto \mathbb{R}$ be such that $\hat{g}(0) = 0$ and $\nabla \hat{g}(0) = 0$ then

$$\sup_{\underline{\xi}, \vec{\eta}} \frac{|H_s[\hat{g}](\underline{\xi}, \vec{\eta})|}{\|\underline{\xi}\|^2 + \|\vec{\eta}\|^2} \leq CM(1 + \Delta_p)d_2(g, 0).$$

Proof. Let $h : \mathbb{R}^3 \mapsto \mathbb{R}$ with $\hat{h}(0) = 0$ and $\nabla \hat{h}(0) = 0$. Then we have

$$\begin{aligned} \left| R[\hat{h}\hat{G}_{\sigma,s}^1](\vec{\xi}, \vec{\eta}) - R[\hat{h}\hat{G}_{\sigma,s}^1](\vec{\xi}, 0)\hat{G}_{\sigma,s}^1(\vec{\eta}) \right| &= \left| \int_{\mathbb{S}^2} \hat{h}(\vec{\xi}^*) \left(\hat{G}_{\sigma,s}^1(\vec{\eta}^*) - \hat{G}_{\sigma,s}^1(0^*)\hat{G}_{\sigma,s}^1(\vec{\eta}) \right) d\omega \right| \leq \\ &2 \int_{\mathbb{S}^2} |\hat{h}(\vec{\xi}^*)| \leq \frac{2}{3}d_2(h, 0)\|\vec{\xi}\|^2 \end{aligned}$$

while, using (9), we get

$$\begin{aligned} |\hat{G}_{\sigma,s}^1(\vec{\eta})\langle \vec{\eta}, \vec{m}_S(s) - \vec{m}_\sigma(s) \rangle \hat{h}(\vec{\xi})| &\leq \hat{\Gamma}_{\tau_\sigma(s)}(\vec{\eta})\|\vec{\eta}\|\|\vec{m}_S(s) - \vec{m}_\sigma(s)\|\|\hat{h}(\vec{\xi})\| \leq \\ &(C\tau_\sigma(s) + \|\vec{m}_S(s) - \vec{m}_\sigma(s)\|^2)d_2(h, 0)\|\vec{\xi}\|^2. \end{aligned}$$

The thesis now follows. $\square$

In order to control (I) and (II), it will also be necessary for us to see that $\Phi_{t,s}$ is a contraction in d_2 , which is the content of the following proposition, which is itself based on the analogous argument for the thermostat defined in [13].

Proposition 3. For any $s \leq t$ and for any $f, g : \mathbb{R}^{3M} \rightarrow \mathbb{R}$ such that $\hat{f}(0) = \hat{g}(0)$ and $\nabla \hat{f}(0) = \nabla \hat{g}(0)$, we have

$$d_2(\Phi_{t,s}f, \Phi_{t,s}g) \leq e^{-\frac{1}{3}(t-s)}d_2(f, g).$$

Proof. From (26) together with Lemma A1 and equation (75) in [13] we get that for any $t > 0$

$$d_2((\Psi(t) + \Lambda \text{Id})f, (\Psi(t) + \Lambda \text{Id})g) \leq \left(\Lambda - \frac{1}{3}\right) d_2(f, g)$$

where $\Lambda = M(2 + \lambda_S) + 2N\lambda_R$. Moreover, it is easily verified that, for τ small and $t \geq s$,

$$\Phi_{t+\tau,s}f - \Phi_{t+\tau,s}g = \tau (\Psi(t) + \Lambda \text{Id}) \Phi_{t,s}(f - g) + (1 - \Lambda\tau + O(\tau^2))\Phi_{t,s}(f - g).$$

Thus we have, for τ small enough and $t > s$

$$\begin{aligned} \frac{d}{dt} d_2(\Phi_{t,s}f, \Phi_{t,s}g) &\leq d_2((\Psi(t) + \Lambda \text{Id}) \Phi_{t,s}f, (\Psi(t) + \Lambda \text{Id}) \Phi_{t,s}g) - \Lambda d_2(\Phi_{t,s}f, \Phi_{t,s}g) \\ &\leq -\frac{1}{3} d_2(\Phi_{t,s}f, \Phi_{t,s}g). \end{aligned}$$

The conclusion now follows via Grönwall's lemma. $\square$

Combining Proposition 3, Proposition 2 and (3) immediately yields

$$(I) \leq CM(1 + \Delta_p) e^{-\frac{s}{3}} d_2(f_0, h_0) = CM(1 + \Delta_p) e^{-\frac{s}{3}} d_2(\tilde{f}_0, \Gamma_{\tau_\infty}^M).$$

Next, we consider (II). We first apply Proposition 2 to obtain

$$(II) \leq CM(1 + \Delta_p) d_2(h_s, \Gamma_{\infty,s}^M). \quad (37)$$

We will now use the Duhamel formula to estimate $d_2(h_s, \Gamma_{\infty,s}^M)$. This will require an understanding of $\frac{d}{ds} \Phi_{t,s}$.

Proposition 4. *For any $t \geq s$ we have*

$$\frac{d}{ds} \Phi_{t,s} = -\Phi_{t,s} \Psi(s). \quad (38)$$

Proof. From the group properties of Φ we get, for τ small,

$$\Phi_{t,s+s'} - \Phi_{t,s} = \Phi_{t,s+s'}(\text{Id} - \Phi_{s+s',s}).$$

Dividing by s' and taking the limit for $s' \rightarrow 0$ obtain (38). $\square$

With (38) in hand, we can apply the Duhamel formula and appeal to Proposition 3 to obtain

$$\begin{aligned} d_2(h_s, \Gamma_{\infty,s}^M) &\leq \int_0^s d_2\left(\Phi_{s,s'}\Psi(s')\Gamma_{\infty,s'}^M, \Phi_{s,s'}\frac{d}{ds'}\Gamma_{\infty,s'}^M\right) ds' \\ &\leq C \int_0^s e^{-\frac{1}{3}(s-s')} d_2\left(\Psi(s')\Gamma_{\infty,s'}^M, \frac{d}{ds'}\Gamma_{\infty,s'}^M\right) ds'. \end{aligned}$$

We now turn our attention to estimating this term.

Proposition 5. For all $s' \geq 0$ we have

$$d_2 \left(\Psi(s') \Gamma_{\infty, s'}^M, \frac{d}{ds'} \Gamma_{\infty, s'}^M \right) \leq \frac{1}{6} \sum_{\sigma \in \{+, -\}} (|\tau_\sigma(s') - \tau_\infty| + 2 \|\vec{\delta}_\sigma(s')\|^2)$$

Proof. We begin by observing that

$$\begin{aligned} \widehat{\Psi}(s') \widehat{\Gamma}_{\infty, s'}^M(\underline{\xi}) - \widehat{\frac{d}{ds'} \Gamma_{\infty, s'}^M}(\underline{\xi}) = \\ \sum_{k=1}^M \Gamma_{\infty, s'}^{M-1}(\underline{\xi}^k) \left(\int_{\mathbb{S}^2} \left(e^{i\langle \vec{\delta}_+(s'), \vec{0}_k^* \rangle} \widehat{\Gamma}_{\tau_+(s')}(\vec{0}_k^*) + e^{i\langle \vec{\delta}_-(s'), \vec{0}_k^* \rangle} \widehat{\Gamma}_{\tau_-(s')}(\vec{0}_k^*) \right) \widehat{\Gamma}_\infty(\vec{\xi}_k^*) - \right. \\ \left. 2\widehat{\Gamma}_\infty(\vec{\xi}_k) - i\langle \vec{m}_S(s'), \vec{\xi}_k \rangle \Gamma_\infty(\vec{\xi}_k) \right) =: \sum_{k=1}^M \widehat{\Gamma}_{\infty, s'}^{M-1}(\underline{\xi}^k) D_2(\vec{\xi}_k) \end{aligned}$$

where $0_k^* = \langle \omega, \vec{\xi}_k \rangle \omega$. It will suffice to understand $D_2(\vec{\xi})$. We begin by writing $\widehat{\Gamma}_{\tau_\sigma(s')}(0^*) = \widehat{\Gamma}_{\tau_\sigma(s')}(0^*) - \widehat{\Gamma}_\infty(0^*) + \widehat{\Gamma}_\infty(0^*)$ to obtain

$$\begin{aligned} |D_2(\vec{\xi})| &= \left| \int_{\mathbb{S}^2} e^{i\langle \vec{\delta}_+(s'), \vec{0}^* \rangle} (\widehat{\Gamma}_{\tau_+(s')}(\vec{0}^*) - \widehat{\Gamma}_\infty(0^*)) \widehat{\Gamma}_\infty(\vec{\xi}) d\omega + \int_{\mathbb{S}^2} e^{i\langle \vec{\delta}_-(s'), \vec{0}^* \rangle} \widehat{\Gamma}_\infty(\vec{\xi}) d\omega \right. \\ &\quad + \int_{\mathbb{S}^2} e^{i\langle \vec{\delta}_-(s'), \vec{0}^* \rangle} (\widehat{\Gamma}_{\tau_-(s')}(\vec{0}^*) - \widehat{\Gamma}_\infty(0^*)) \widehat{\Gamma}_\infty(\vec{\xi}) d\omega + \int_{\mathbb{S}^2} e^{i\langle \vec{\delta}_-(s'), \vec{0}^* \rangle} \widehat{\Gamma}_\infty(\vec{\xi}) d\omega \\ &\quad \left. - 2\widehat{\Gamma}_\infty(\vec{\xi}) - i\langle \vec{m}_S(s'), \vec{\xi} \rangle \widehat{\Gamma}_\infty(\vec{\xi}) \right| \\ &\leq \int_{\mathbb{S}^2} \left| \widehat{\Gamma}_{\tau_+(s')}(\vec{0}^*) - \widehat{\Gamma}_\infty(0^*) \right| d\omega + \int_{\mathbb{S}^2} \left| \widehat{\Gamma}_{\tau_-(s')}(\vec{0}^*) - \widehat{\Gamma}_\infty(0^*) \right| d\omega \\ &\quad + \left| \int_{\mathbb{S}^2} e^{i\langle \vec{\delta}_+(s'), \vec{0}^* \rangle} + e^{i\langle \vec{\delta}_-(s'), \vec{0}^* \rangle} d\omega - 2 - i\langle \vec{m}_S(s'), \vec{\xi} \rangle \right| \end{aligned}$$

Using that $\int_{\mathbb{S}^2} \|0^*\|^2 d\omega = \frac{1}{3} \|\vec{\xi}\|^2$, the first term in this estimate satisfies

$$\int_{\mathbb{S}^2} \left| \widehat{\Gamma}_{\tau_\sigma(s')}(\vec{0}^*) - \widehat{\Gamma}_\infty(0^*) \right| d\omega \leq \frac{1}{3} d_2(\Gamma_{\tau_\sigma(s')}, \Gamma_\infty) \|\vec{\xi}\|^2 \leq \frac{1}{6} |\tau_\sigma(s') - \tau_\infty| \|\vec{\xi}\|^2.$$

We can control the second term by expanding $e^{i\langle \vec{\delta}_\sigma(s'), \vec{0}^* \rangle}$ using

$$e^x = 1 + x + x^2 \int_0^1 e^{tx} (1-t) dt$$

and using (9) to obtain

$$\begin{aligned}
& \left| \int_{\mathbb{S}^2} e^{i\langle \vec{\delta}_+(s'), \vec{0}^* \rangle} + e^{i\langle \vec{\delta}_-(s'), \vec{0}^* \rangle} - 2 - i\langle \vec{m}_S, \xi \rangle \right| \\
&= \left| \int_{\mathbb{S}^2} \langle \vec{\delta}_+(s'), \vec{0}^* \rangle^2 \int_0^1 e^{i\langle \vec{\delta}_+(s'), \vec{0}^* \rangle t} (1-t) dt d\omega + \int_{\mathbb{S}^2} \langle \vec{\delta}_-(s'), \vec{0}^* \rangle^2 \int_0^1 e^{i\langle \vec{\delta}_-(s'), \vec{0}^* \rangle t} (1-t) dt d\omega \right| \\
&\leq \int_{\mathbb{S}^2} \langle \vec{\delta}_+(s'), \vec{0}^* \rangle^2 + \langle \vec{\delta}_-(s'), \vec{0}^* \rangle^2 d\omega \leq \frac{1}{3} \left(\|\vec{\delta}_+(s')\|^2 + \|\vec{\delta}_-(s')\|^2 \right) \|\xi\|^2
\end{aligned}$$

Combining these estimates we finally get the thesis. $\square$

Proposition 5, Proposition 1 and (37) now yield

$$(II) \leq CM(1 + \Delta_p)(\Delta_p + \Delta_T)e^{-\frac{2Ms}{3N}}.$$

It remains for us to understand (III).

Proposition 6. *For any $s \geq 0$ we have*

$$(III) \leq CM \sum_{\sigma} \left(|\tau_{\sigma}(s) - \tau_{\infty}| + \|\vec{\delta}_{\sigma}(s')\|^2 \right)$$

Proof. The proof is very similar to that for Proposition 5. We refer readers to Appendix B for full details. $\square$

Combining the estimates for (I), (II), and (III), we obtain

$$D_1 \leq CM(1 + \Delta_p) \left(e^{-\frac{s}{3}} d_2(\tilde{f}_0, h_0) + (\Delta_T + \Delta_p) e^{-\frac{2Ms}{3N}} \right) \quad (39)$$

and thus, taking into account (28) and Proposition 1, we finally get

$$d_2(F_t, \bar{F}_t) \leq C \frac{M}{\sqrt{N}} (1 + \Delta_p) E_4(f)^{\frac{5}{6}} \left(d_2(\tilde{f}_0, \Gamma_{\tau_{\infty}}^M)^{\frac{1}{6}} \left(1 - e^{-\frac{t}{18}} \right) + \frac{9N}{M} \left(1 - e^{-\frac{M}{9N}t} \right) (\Delta_p + \Delta_T)^{\frac{1}{6}} \right) \quad (40)$$

This concludes the proof of Theorem 1.

To obtain Corollary 2 we observe that, if $T_+ = T_- = T$ and $\vec{p}_+ = \vec{p}_-$ then $\tau_+(t) = \tau_-(t) =: \tau(t)$ and $\vec{m}_+(t) = \vec{m}_-(t) =: \vec{m}(t)$ for every $t \geq 0$. The IVP (9) now reads

$$\frac{d}{dt} \begin{pmatrix} \vec{m}_S \\ \vec{m} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 & 2 \\ \frac{M}{N} & -\frac{M}{N} \end{pmatrix} \begin{pmatrix} \vec{m}_S \\ \vec{m} \end{pmatrix}, \quad \begin{pmatrix} \vec{m}_S(0) \\ \vec{m}(0) \end{pmatrix} = \begin{pmatrix} \vec{p}_S \\ \vec{p} \end{pmatrix}$$

from which we get

$$\vec{m}(t) - \vec{m}_S(t) = e^{-\frac{1}{3}(2 + \frac{M}{N})t} (\vec{p} - \vec{p}_S).$$

With a similar argument we get

$$|\tau(s) - \tau_S(s)| \leq e^{-\frac{1}{3}(2 + \frac{M}{N})t} \left(\Delta_T + \frac{1}{3}\Delta_p \right).$$

Using again the estimates for (I), (II), and (III) conclude the proof.

Declarations

Conflict of Interest

The authors have no conflicts of interest to disclose.

Author Contributions

Federico Bonetto: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).

Matthew Powell: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

- [1] Mark Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III*, pages 171–197, Berkeley and Los Angeles, 1956. University of California Press.
- [2] Eric Carlen, M. C. Carvalho, and Michael Loss. Many-body aspects of approach to equilibrium. In *Journées “Équations aux Dérivées Partielles” (La Chapelle sur Erdre, 2000)*, pages Exp. No. XI, 12. Univ. Nantes, Nantes, 2000.
- [3] E. A. Carlen, M. C. Carvalho, and M. Loss. Determination of the spectral gap for Kac’s master equation and related stochastic evolution. *Acta Math.*, 191(1):1–54, 2003.
- [4] Natesh S. Pillai and Aaron Smith. On the mixing time of Kac’s walk and other high-dimensional Gibbs samplers with constraints. *The Annals of Probability*, 46(4):2345 – 2399, 2018.
- [5] H. Tossounian. Equilibration in the Kac model using the gtw metric d_2 . *Journ, Stat. Phys.*, 169:168–186, 2017.
- [6] Stéphane Mischler and Clément Mouhot. Kac’s program in kinetic theory. *Invent. Math.*, 193(1):1–147, 2013.
- [7] Stéphane Mischler and Clément Mouhot. About Kac’s program in kinetic theory. *C. R. Math. Acad. Sci. Paris*, 349(23-24):1245–1250, 2011.
- [8] Alain Solt Sznitman. Équations de type de boltzmann, spatialement homogènes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 66:559–592, 1984.

- [9] G.A. Bird. *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*. Clarendon Press Publication, 1994.
- [10] Federico Bonetto, Michael Loss, and Ranjini Vaidyanathan. The Kac model coupled to a thermostat. *J. Stat. Phys.*, 156(4):647–667, 2014.
- [11] Federico Bonetto, Michael Loss, Hagop Tossounian, and Ranjini Vaidyanathan. Uniform approximation of a Maxwellian thermostat by finite reservoirs. *Comm. Math. Phys.*, 351(1):311–339, 2017.
- [12] Federico Bonetto, Alissa Geisinger, Michael Loss, and Tobias Ried. Entropy decay for the Kac evolution. *Comm. Math. Phys.*, 363(3):847–875, 2018.
- [13] Federico Bonetto, Michael Loss, and Matthew Powell. A kac system interacting with two heat reservoirs. Submitted, 2025.
- [14] Gabetta, G. Toscani, and B. Wennberg. Metrics for probability distributions and the trend to equilibrium for solutions of the boltzmann equation. *Journ, Stat. Phys.*, 81:901–934, 1995.
- [15] P. Bonetto, F. Cheng, A. Popa, M. Powell, and S Tung. A kac system ... In preparation, 2026.
- [16] Eric Carlen, Joel Lebowitz, and Clement Mouhot. Exponential approach to, and properties of, a non-equilibrium steady state in a dilute gas. *Brazilian Journal of Probability and Statistics*, 29(2):372–386, 2015.

A Estimating moments of F_s

A key observation from [13] was that, in the standard static thermostat case, there is a constant C independent of M such that $\sup_s \|H_s[\hat{f}_s]\|_{3,1} \leq CM$, which allowed us to obtain a uniform estimate in the one thermostat case and a ‘reasonable’ short-time estimate in the two thermostat case. We will see, by a similar argument, that the same is true for the dynamic thermostats considered here.

Proposition 7. *There exists a constant C such that $\sup_s \|H_s[\hat{f}_s]\|_{3,1} \leq CME_4(f_0)$.*

Reasoning like in Lemmas B2 and B3 in [13] we can show that

$$\|H_s[\hat{f}_s]\|_{3,1} \leq CME_4(f_s)$$

We thus have to study the evolution of the moments of f_s . To do this, we will define an adjoint evolution on the space of polynomials in $\underline{v}$. The properties of this evolution will show that $E_4(f_s) \leq CE_4(f_0)$ that proof Proposition 7.

Given $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$, it is easy to see that

$$\int g(\vec{v})B(t)[f](\vec{v})d\vec{v} = \int B^\dagger(t)[g](\vec{v})f(\vec{v})d\vec{v}$$

where $B(t)$ is given by (25) while $B^\dagger(t)$ is the adjoint of $B(t)$ and is given by

$$B^\dagger(t)[g](\vec{v}) = \int_{\mathbb{R}^3} \mathcal{M}_t(\vec{u}) \int_{\mathbb{S}^2} g(\vec{v} - (\vec{u} - \vec{v}, \omega)\omega) d\omega d\vec{u}.$$

Observe that, if $p(\vec{v})$ is a polynomial of degree K , then $p(\vec{v} - (\vec{u} - \vec{v}, \omega)\omega)$ can be seen as a polynomial in $\vec{v}$ and $\vec{u}$ with coefficients depending on ω . It follows that $B^\dagger(t)[p](\vec{v})$ is a polynomial in $\vec{v}$ of degree $\leq K$ whose coefficients depend on the moments of $\mathcal{M}(t)$. Similarly we can define the adjoint $R^\dagger$ of R acting on function $f : \mathbb{R}^6 \mapsto \mathbb{R}$. In this case it turns out that $R^\dagger = R$. Moreover, for a similar reason, R maps the space of homogeneous polynomials of degree K into itself.

We thus start by describing the action of $\mathcal{L}_S$ and $\mathcal{L}_{B(t)}^\dagger$ on polynomials of degree at most K . We consider the space W_o^K of homogeneous polynomials of degree K in $\vec{v}$ with the basis formed by the $\vec{v}^{\vec{\alpha}}$ with $\|\vec{\alpha}\|_1 = K$ and the space $W^K = \bigoplus_{k=0}^K W_o^k$ of polynomials of degree $\leq K$ together with the natural projection Q_k from W^K to W_o^k for $k \leq K$. Analogously we consider the space U_o^K of symmetric homogeneous polynomials of degree K in $\vec{v}_1$ and $\vec{v}_2$ with the basis formed by the $\vec{v}_1^{\vec{\alpha}_1} \vec{v}_2^{\vec{\alpha}_2} + \vec{v}_1^{\vec{\alpha}_2} \vec{v}_2^{\vec{\alpha}_1}$ with $\|\alpha\|_1 + \|\alpha\|_2 = K$ and the space $U^K = \bigoplus_{k=0}^K U_o^k$ of symmetric polynomials of degree $\leq K$.

We can now summarize the above discussion, together with a few observations from [13, 15]:

1. For every k , $B^\dagger(t)W_o^k \subset W^k$, while $RU_o^k \subset U_o^k$.
2. $B^k := Q_k B^\dagger(t)|_{W_o^k}$ is independent of $\mathcal{M}(t)$ and thus of t .
3. $R - I$ acting on U_o^k is negative semi-definite while, for $k > 0$, $B^k - I$ is negative definite¹.

Observe that point 1 means that $Q_k B^\dagger(t)|_{W_o^l} = 0$ if $k < l$, that is $B^\dagger(t)$ is block upper triangular w.r.t. the W_o^k . Similarly R is block diagonal w.r.t. the U_o^k .

We now return to polynomials in $\underline{v}$. For every $\underline{\alpha} \in (\mathbb{N}^3)^M$ with $\sum_i \|\vec{\alpha}_i\|_1 = K$ we can associate the degree K monomial $\underline{v}^{\underline{\alpha}} = \prod_i \vec{v}_i^{\vec{\alpha}_i}$. Let $\underline{n}_\alpha : \mathbb{N}^3 / \{0, 0, 0\} \mapsto \mathbb{N}$ be the *occupation numbers* of $\underline{\alpha}$, that is $\underline{n}_\alpha(\vec{\beta}) = \#\{i \mid \vec{\alpha}_i = \vec{\beta}\}$. Then the polynomials

$$e_{\underline{n}}(\underline{v}) = \mathcal{S}(\underline{v}^{\underline{\alpha}}) \quad \text{with} \quad \underline{n}_\alpha = \underline{n}$$

and where $\mathcal{S}$ is the symmetrization operator:

$$\mathcal{S}[p](\underline{v}) := \frac{1}{M!} \sum_{\pi} p(\vec{v}_{\pi(1)}, \dots, \vec{v}_{\pi(M)}).$$

form a basis in the space V_o^K of homogeneous polynomials of degree K in $\underline{v}$. Let moreover $V^K = \bigoplus_{k=0}^K V_o^k$ the space of polynomials in $\underline{v}$ of degree at most K and P_k the natural projection from V^K to V_o^k for $k \leq K$.

¹We say that a matrix A is negative definite if $A + A^T$ has all negative eigenvalues. In our case the transpose are taken w.r.t. the basis discussed in the previous paragraph. Observe that from [15] it follows that $B^k - I$ and $R - I$ are diagonalizable with all negative eigenvalues.

Claim 1. Let $d_K = \dim(V^K)$ and $L_{B(t)} \in \mathbb{R}^{d_K \times d_K}$ be the matrix representation of $\sum_{i=1}^M (B_i^\dagger(t) - \text{Id})$ in the basis formed by the $e_{\underline{n}}$ with $\sum_{\underline{\alpha}} \underline{n}(\underline{\alpha}) = K$. Analogously let $L_R \in \mathbb{R}^{d_K \times d_K}$ be the matrix representation of $(M-1)^{-1} \sum_{i < j}^M (R_{i,j} - \text{Id})$. We have

1. $L_{B(t)}$ is block upper triangular while L_R is block diagonal w.r.t. the V_o^k .
2. $L_{B(t)}$ does not depend on M while $L_R = \bar{L}_R + M^{-1}P$ where $\bar{L}_R$ does not depend on M while P is bounded uniformly in M .
3. $P^k L_{B(t)}|_{V_o^k}$ does not depend on t .
4. $P^k(L_R + L_{B(t)})|_{V_o^k}$, $k > 0$, is negative definite (see footnote 1).

Proof. Points 1, 3, and 4 follow directly from the observations above. Regarding point 3 we observe that

$$\mathcal{L}_{B(t)}^\dagger e_{\underline{n}}(\underline{v}) = \mathcal{S}(\mathcal{L}_{B(t)}^\dagger \underline{v}^\alpha) = \mathcal{S} \left(\sum_{i=1}^M B_i^\dagger(t) [\vec{v}_i^{\vec{\alpha}_i}] \prod_{k \neq i} \vec{v}_k^{\vec{\alpha}_k} \right)$$

for any $\underline{\alpha}$ such that $\underline{n}_{\underline{\alpha}} = \underline{n}$. Let $\#(\underline{\alpha})$ be the number of $\vec{\alpha}_i \neq (0,0,0)$ in $\underline{\alpha}$. Since $\#(\underline{\alpha}) \leq K < M$, given $\underline{n}$ we can chose $\underline{\alpha}$ such that $\vec{\alpha}_i \neq 0$ for $i \leq \#(\underline{\alpha})$ while $\vec{\alpha}_i = 0$ for $i > \#(\underline{\alpha})$. We thus get

$$\mathcal{L}_{B(t)}^\dagger e_{\underline{n}}(\underline{v}) = \mathcal{S} \left(\sum_{i=1}^{\#(\underline{\alpha})} B_i^\dagger(t) [\vec{v}_i^{\vec{\alpha}_i}] \prod_{k \neq i} \vec{v}_k^{\vec{\alpha}_k} \right)$$

that clearly does not depend on M . Similarly we can write

$$\mathcal{L}_S e_{\underline{n}}(\underline{v}) = \mathcal{S} \left(\frac{1}{M-1} \sum_{i < j} R_{i,j} [\vec{v}_i^{\vec{\alpha}_i} \vec{v}_j^{\vec{\alpha}_j}] \prod_{k \neq i,j} \vec{v}_k^{\vec{\alpha}_k} \right) = \mathcal{S} \left(\sum_{i \leq \# \underline{\alpha}} R_{i,M} [\vec{v}_i^{\vec{\alpha}_i}] \prod_{k \neq i, k \leq \# \underline{\alpha}} \vec{v}_k^{\vec{\alpha}_k} \right) + O \left(\frac{1}{M} \right).$$

□

We can now return to the evolution of the moments of f_s . We get

$$\int \underline{v}^\alpha f_s(\underline{v}) d\underline{v} = \int \Phi_{s,0}^\dagger[e_{\underline{n}_{\underline{\alpha}}}] (\vec{v}) f_0(\underline{v}) d\underline{v} = \sum_{\underline{n}} m_{\underline{n}} A(t)_{\underline{n}, \underline{n}_{\underline{\alpha}}}$$

where

$$m_{\underline{n}} = \int e_{\underline{n}}(\underline{v}) f_0(\underline{v}) d\underline{v}$$

while $A(t)$ solves the equation

$$\dot{A}(t) = A(t)(L_S + L_{B(t)}^\dagger); \quad A(0) = \text{Id}.$$

We thus get

$$E_K(f_s) \leq \sqrt{d_K} E_K(f_0) \|A(t)\|_2 .$$

Calling $L^{l,k}(t) = P_k(L_S + L_{B(t)}^\dagger)|_{V_o^l}$ we get $L^{k,l} = 0$, for $k < l$ so that

$$\begin{aligned} P_K A(t) &= e^{L^{K,K}t} P_K \\ P_{K-1} A(t) &= e^{L^{K-1,K-1}t} P_K + \int_0^s e^{L^{K-1,K-1}(t-s)} L^{K-1,K}(t) P_K A(t) \\ &\vdots \\ P_k A(t) &= e^{L^{k,k}t} P_k + \sum_{l=k+1}^K \int_0^s e^{L^{k,k}(t-s)} L^{l,k}(t) P_k A(t) . \end{aligned} \tag{41}$$

The thesis now follow easily by solving recursively (41) and observing that, from point 2 and 4 of Claim 1 and the continuity of eigenvalues and eigenvectors, it follows that $\|e^{L^{k,k}t}\|_2 \leq C e^{-ct}$, for suitable constant $C, c > 0$.

B Proofs from Subsection 3.2

We collect here the proofs of some of the propositions from Subsection 3.2.

Proof of Proposition 1. From (9) we see that $\vec{\delta}_\sigma$ satisfy

$$\frac{d}{dt} \begin{pmatrix} \vec{\delta}_+ \\ \vec{\delta}_- \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 + \frac{M}{N} & -1 \\ -1 & 1 + \frac{M}{N} \end{pmatrix} \begin{pmatrix} \vec{\delta}_+ \\ \vec{\delta}_- \end{pmatrix}$$

so that we get

$$\begin{aligned} \vec{\delta}_+(s) + \vec{\delta}_-(s) &= e^{-\frac{s}{3}(2 + \frac{M}{N})} (\vec{\delta}_+(0) + \vec{\delta}_-(0)) \\ \vec{\delta}_+(s) - \vec{\delta}_-(s) &= e^{-\frac{s}{3}\frac{M}{N}} (\vec{\delta}_+(0) - \vec{\delta}_-(0)) . \end{aligned}$$

Equation (34) now follows using that

$$\|\vec{\delta}_+\|^2 + \|\vec{\delta}_-\|^2 = \frac{1}{2} \left(\|\vec{\delta}_+ + \vec{\delta}_-\|^2 + \|\vec{\delta}_+ - \vec{\delta}_-\|^2 \right) .$$

Similarly we get

$$\frac{d}{dt} \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 + \frac{M}{N} & -1 \\ -1 & 1 + \frac{M}{N} \end{pmatrix} \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} - \frac{1}{9} \begin{pmatrix} 1 - \frac{M}{N} & 1 \\ 1 & 1 - \frac{M}{N} \end{pmatrix} \begin{pmatrix} \|\vec{\delta}_+\|^2 \\ \|\vec{\delta}_-\|^2 \end{pmatrix}$$

so that

$$\begin{aligned}\epsilon_+(s) + \epsilon_-(s) &= e^{-\frac{s}{3}(2+\frac{M}{N})}(\epsilon_+(0) + \epsilon_-(0)) - \frac{1}{9} \left(2 - \frac{M}{N}\right) \int_0^s e^{-\frac{s-s'}{3}(2+\frac{M}{N})} (\|\vec{\delta}_+(s')\|^2 + \|\vec{\delta}_-(s')\|^2) ds' \\ \epsilon_+(s) - \epsilon_-(s) &= e^{-\frac{s}{3}\frac{M}{N}}(\epsilon_+(0) - \epsilon_-(0)) - \frac{M}{9N} \int_0^s e^{-\frac{(s-s')M}{3N}} (\|\vec{\delta}_+(s')\|^2 - \|\vec{\delta}_-(s')\|^2) ds' .\end{aligned}$$

Using that

$$\frac{1}{2}(|\epsilon_+ + \epsilon_-| + |\epsilon_+ - \epsilon_-|) \leq |\epsilon_+| + |\epsilon_-| \leq |\epsilon_+ + \epsilon_-| + |\epsilon_+ - \epsilon_-|$$

we easily get (35).

Finally we observe that, for every s ,

$$N^* \tau_\infty = N \tau_+(s) + N \tau_-(s) + M \tau_S(s) + N \|\vec{m}_+(s) - \vec{m}_S(s)\|^2 + N \|\vec{m}_-(s) - \vec{m}_S(s)\|^2$$

so that

$$N^*(\tau_+(s) - \tau_\infty) = (N + M)(\tau_+(s) - \tau_S(s)) - N(\tau_-(s) - \tau_S(s)) + N(\|\vec{\delta}_+(s)\|^2 + \|\vec{\delta}_-(s)\|^2)$$

from which (36) follows. $\square$

Proof of Proposition 6. For convenience of notation, we begin by writing

$$\Theta_{\sigma,t}(x, y) := e^{i\langle x, \vec{m}_s(t) \rangle} \cdot e^{i\langle y, \vec{m}_\sigma(t) \rangle}.$$

By recalling the definition of $H_s[\widehat{\Gamma}_{\infty,s}^M](\underline{\xi}, \vec{\eta}_1)$ and proceeding as in Proposition 5, we can write

$$H_s[\widehat{\Gamma}_{\infty,s}^M](\underline{\xi}, \vec{\eta}) = \sum_{\sigma=\pm} \sum_{k=1}^M \widehat{\Gamma}_{\infty,s}^{M-1}(\underline{\xi}^k) G_{\sigma',s}(\vec{\eta}_{\sigma'}) D_{3,\sigma}(\vec{\xi}_k, \vec{\eta}_\sigma)$$

where

$$\begin{aligned}D_{3,\sigma}(\vec{\xi}, \vec{\eta}) &= \int_{\mathbb{S}^2} \Theta_{\sigma,s}(\vec{\xi}^*, \vec{\eta}^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}^*) - \Theta_{\sigma,s}(\vec{\xi}^*, 0^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \widehat{\Gamma}_{\sigma,s}(0^*) \Theta_{\sigma,s}(0, \vec{\eta}) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) d\omega \\ &\quad - i \frac{1}{3} \Theta_{\sigma,s}(\vec{\xi}, \vec{\eta}) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) \widehat{\Gamma}_\infty(\vec{\xi}) \langle \vec{\eta}, \vec{m}_S(s) - \vec{m}_\sigma(s) \rangle\end{aligned}$$

By writing $\widehat{\Gamma}_{\sigma,s}(\vec{\eta}) = \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) - \widehat{\Gamma}_\infty(\vec{\eta}) + \widehat{\Gamma}_\infty(\vec{\eta})$ we have

$$\begin{aligned}\Theta_{\sigma,s}(\vec{\xi}^*, \vec{\eta}^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}^*) - \Theta_{\sigma,s}(\vec{\xi}^*, 0^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \widehat{\Gamma}_{\sigma,s}(0^*) \Theta_{\sigma,s}(0, \vec{\eta}) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) \\ = \Theta_{\sigma,s}(\vec{\xi}^*, \vec{\eta}^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \left(\widehat{\Gamma}_{\sigma,s}(\vec{\eta}^*) - \widehat{\Gamma}_\infty(\vec{\eta}^*) \right) \\ - \Theta_{\sigma,s}(\vec{\xi}^*, 0^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \left(\widehat{\Gamma}_{\sigma,s}(0^*) - \widehat{\Gamma}_\infty(0^*) \right) \Theta_{\sigma,s}(0, \vec{\eta}) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) \\ + \widehat{\Gamma}_\infty(\vec{\xi}) \left(\Theta_{\sigma,s}(\vec{\xi}^*, \vec{\eta}^*) \widehat{\Gamma}_\infty(\vec{\eta}) - \Theta_{\sigma,s}(\vec{\xi}^*, 0^*) \Theta_{\sigma,s}(0, \vec{\eta}) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) \right) .\end{aligned} \tag{42}$$

Now observe that

$$\left| \Theta_{\sigma,s}(\vec{\xi}^*, \vec{\eta}^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \left(\widehat{\Gamma}_{\sigma,s}(\vec{\eta}^*) - \widehat{\Gamma}_\infty(\vec{\eta}^*) \right) \right| \leq \|\vec{\eta}^*\|^2 d_2(\Gamma_{\sigma,s}, \Gamma_\infty) = \frac{1}{2} \|\vec{\eta}^*\|^2 |\tau_\sigma(s) - \tau_\infty| \quad (43)$$

and similarly

$$\left| \Theta_{\sigma,s}(\vec{\xi}^*, 0^*) \widehat{\Gamma}_\infty(\vec{\xi}^*) \left(\widehat{\Gamma}_{\sigma,s}(0^*) - \widehat{\Gamma}_\infty(0^*) \right) \widehat{\Gamma}_{\sigma,s}(\vec{\eta}) \Theta_{\sigma,s}(0, \vec{\eta}) \right| = \frac{1}{2} \|0^*\|^2 |\tau_\sigma(s) - \tau_\infty|. \quad (44)$$

Meanwhile, writing

$$\Theta_{\sigma,s}(x, y) = 1 + i \langle x, \vec{m}_s(t) \rangle + i \langle y, \vec{m}_\sigma(t) \rangle - (\langle x, \vec{m}_s(t) \rangle + \langle y, \vec{m}_\sigma(t) \rangle)^2 \int_0^1 \Theta_{\sigma,s}(tx, ty)(1-t) dt$$

and setting $\vec{x} = \langle \omega, \vec{\xi} \rangle \omega$ and $\vec{y} = \langle \omega, \vec{\eta} - \vec{\xi} \rangle \omega$, yields

$$\begin{aligned} |D_{3,\sigma}(\vec{\xi}, \vec{\eta})| &= \left| \int_{\mathbb{S}^2} i \langle \omega, \eta \rangle \langle \omega, \vec{m}_S(s) - \vec{m}_\sigma(s) \rangle - \frac{i}{3} \langle \vec{\eta}, \vec{m}_S(s) - \vec{m}_\sigma(s) \rangle \right. \\ &\quad + \langle \omega, \vec{\xi} \rangle^2 \langle \omega, \vec{m}_S(s) - \vec{m}_\sigma(s) \rangle^2 \int_0^1 \Theta_{\sigma,s}(t\vec{x}, -t\vec{x})(1-t) dt \\ &\quad \left. - \langle \omega, \vec{\eta} - \vec{\xi} \rangle^2 \langle \omega, \vec{m}_S(s) - \vec{m}_\sigma(s) \rangle^2 \int_0^1 \Theta_{\sigma,s}((t\vec{y}, -t\vec{y})(1-t) dt d\omega \right| \\ &\leq 4(\|\vec{\xi}\|^2 + \|\vec{\eta}\|^2) \|\vec{m}_S(s) - \vec{m}_\sigma(s)\|^2 \end{aligned} \quad (45)$$

Combining (43), (44), and (45) with (B) and (42) yields

$$\begin{aligned} \frac{|H_s[\hat{f}_{\infty,s}](\underline{\xi}, \vec{\eta}_{\pm,1})|}{\|\underline{\xi}\|^2 + \|\vec{\eta}\|} &\leq \sum_{\sigma} \sum_{j=1}^M \frac{4(\|\underline{\xi}_j\|^2 + \|\vec{\eta}\|^2)(|\tau_\sigma(s) - \tau_\infty| + \|\vec{m}_S(s) - \vec{m}_\sigma(s)\|^2)}{\|\underline{\xi}\|^2 + \|\vec{\eta}\|} \\ &\leq CM \sum_{\sigma} (|\tau_\sigma(s) - \tau_\infty| + \|\vec{m}_S(s) - \vec{m}_\sigma(s)\|^2). \end{aligned}$$

□