The Kac model and (Non-)Equilibrium Statistical Mechanics.

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Plan of the talk.

- Introduction to the physics of gases.
- The Kac model and gas kinetic.
- Classical results on the Kac model.
- Local Perturbation.

1000 particles initially confined in a quarter of the container and with independent velocity uniformly distributed in $[-1, 1]$.

Left panel: particle position. Right panel: histogram of the x-velocity (time smoothed).

Exactly as before but the particles do not collide.

Left panel: particle position. Right panel: histogram of the x-velocity (time smoothed).

A little history: Fourier.

Heat, like gravity, penetrates every substance of the universe, its ray occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics.

— *Théorie analytique de la chaleur, 1822*
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But whatever may be the range of mechanical theories, they do not apply to the effects of heat. These make up a special order of phenomena, which cannot be explained by the principles of motion and equilibria.

— *Ibidem*

Here are some physical quantities for oxygen at ambient condition

- temperature $T = 273\mathrm{K}$
- pressure $P = 1013\mathrm{mbar}$
- number density $\delta = N/V = 2.7 \times 10^{25}\mathrm{molecules/m}^3$
- kinetic radius $r = 1.73 \times 10^{-10}\mathrm{m}$
- molecule average speed $v = 1.58 \times 10^2\mathrm{m/s}$
- mean free path $d = 1.0 \times 10^{-7}\mathrm{m}$
- mean free time $\lambda = 0.6 \times 10^{-5}\mathrm{s}$

Consider our cubic meter of oxygen. How can we describe it when it is not in equilibrium? E.g. when it is stirred around by a paddle or heated from one of his walls.

Macroscopic vs microscopic

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In such a case, we normally use macroscopic equations like the Heat Equation, Navier-Stokes Equation, etc. They describe the evolution of quantities like the local temperature $T(x)$, density $\delta(x)$, velocity $u(x)$, entropy $s(x)$, and more.

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We can consider our cube has made up of a large number of small cubes, say 10^{12} cubes of side $1\mu = 10^{-4}\mathrm{m}$. Each of such cubes will contain in average 10^{13} particles. Thus from the macroscopic point of view each of these cubes is a point and its position is the x appearing in the macroscopic equations. From the microscopic point of view it is an infinite system endowed of temperature, entropy. etc.

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This image is called *local equilibrium* and is sometime expressed saying that *a macroscopic system can be thought as composed by infinitely many volume elements that are macroscopically infinitesimal and microscopically infinite*.

Local equilibrium with 25 volume elements.

Some more history: de Groot and Mazur.

It will now be assumed that, although the total system is not in equilibrium, there exists within small mass elements a state of "local" equilibrium for which the local entropy s is the same function of u, v and c_k as in real equilibrium.

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The hypothesis of "local" equilibrium can, from a macroscopic point of view, only be justified by virtue of the validity of the conclusions derived from it.

— *Ibidem*

Work in collaboration with:

- Michael Loss: School of Math., GaTech
- Ranjini Vaidyanathan: Former graduate student, School of Math., GaTech
- Hagop Tossounian: Former graduate student, School of Math., GaTech. Now in Santiago, Chile.
- Alissa Geisinger: Graduate student, Universität Tübingen
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Publications:

- F. B., M. Loss, R. Vaidyanathan,*The Kac Model Coupled to a Thermostat*, JSP 156:847-667 (2014).
- H. Tossounian, R. Vaidyanathan:*Partially Thermostated Kac Model*, JMP 56 (2015).
- F. B., M. Loss, H. Tossounian, R. Vaidyanathan, *Uniform Approximation of a Maxwellian Thermostat by Finite Reservoirs*, CMP 351 (2017).
- H. Tossounian, *Equilibration in the Kac Model using the GTW Metric d_2*, JSP 169 (2017).
- F. B., A. Geisiger, M. Loss, T. Ried, *Entropy Decay for the Kac Evolution*, CMP 353 (2019)

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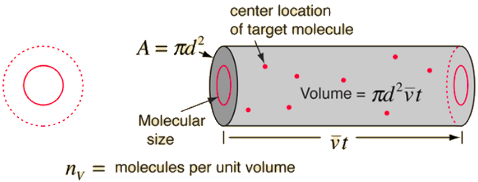
2. particles are hard spheres of small radius r;
3. collisions are elastic;

Mean free time.

The number of collision ν a particle suffers in a time t is:

$$\nu = \pi d^2 \, \bar{v} t \; M/V$$



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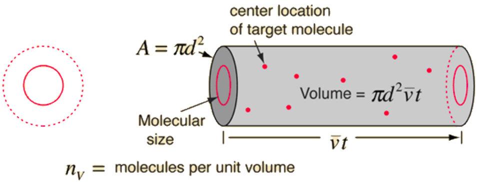

Since

$$\bar{v} \simeq \sqrt{\frac{3k_B T}{m}}$$

it is reasonable define the Grad-Boltzman limit as

$$r \to 0 \quad , \quad M \to \infty \quad \text{such that} \quad \pi r^2 \sqrt{\frac{3k_B T}{m}} M/V \to \lambda^{-1}$$

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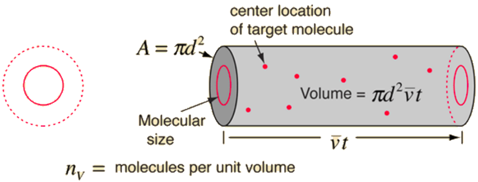

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λ is the natural time scale for the system. Fix unit of time such that $\lambda = 1$.

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λ_M is fixed in such a way that the average time between two collision of a given particle is independent of M. That is $\lambda_M = 1/(M-1)$.

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2. Energy and momentum are redistributed randomly.
3. the collision rate between two particles does not depend on their velocities. This are often called “Maxwellian Molecules”.

State of the system

$$F(V):\mathbb{R}^M\to\mathbb{R}\qquad\qquad V=(v_1,v_2,\dots,v_M)\in\mathbb{R}^M,$$

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If F is the state of the system before particle i and j collide, just after the collision the state is

$$R_{i,j}F(V)=\int\rho(\theta)F(r_{i,j}(\theta)V)d\theta$$

where

$$r_{1,2}(\theta)V=(v_1\cos(\theta)-v_2\sin(\theta),v_1\sin(\theta)+v_2\cos(\theta),v_3,\dots)$$

that is, $r_{i,j}(\theta)$ is a rotation of angle θ in the i, j plane.

The evolution.

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We need

$$\int \rho(\theta) \sin\theta \cos\theta \mathrm{d}\theta = 0 \ .$$

but for most of the talk we will assume

$$\rho(\theta) = \frac{1}{2\pi}$$

The evolution.

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so that the evolution is given by

$$F_t = e^{-Mt} \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k F_0 = e^{\mathcal{L}_S t} F_0$$

where

$$\mathcal{L}_S = M(Q - I) = \frac{2}{M-1} \sum_{i<j} (R_{i,j} - I).$$

Thus F_t satisfies the equation:

$$\dot{F}_t = \mathcal{L}_S F_t = M(QF - F)$$

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Master Equation.

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Given an initial distribution $F(V)$, the evolution brings it toward its projection on the rotationally invariant distributions, that is toward

$$F_R(V) = \int_{S^{M-1}} F(|V|\omega)d\sigma(\omega)$$

where $d\sigma(\omega)$ the normalized volume measure on the unit sphere S^{M-1}.

Convergence to equilibrium.

Carlen-Carvalho-Loss (2000) showed that

$$\left\| e^{t\mathcal{L}_S} F(V) - F_R(V) \right\|_2 \leq C e^{-L^{(1)} t}$$

where $\| \cdot \|_2$ is the $L^2(\mathbb{R}^M)$ norm and

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The L^2 norm has one major problem. Assume that

$$F(V) = \prod_{i=1}^{M} f(v_i) \qquad \text{and} \qquad G(V) = \prod_{i=1}^{M} g(v_i)$$

then

$$\|F - G\|_2 \simeq C^M \|h - g\| \qquad \text{with} \qquad C > 1.$$

Convergence to equilibrium in entropy.

The entropy with respect to the steady state is defined as

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In general

$$S(F|F_R) \geq 0 \qquad\qquad S(F|F_R) = 0 \quad \Leftrightarrow \quad F = F_R$$

and

$$\dot{S}(F_t|F_R) \leq 0$$

and

$$F(V) = \prod_{i=1}^{M} f(v_i) \qquad \Rightarrow \qquad S(F|F_R) = O(M).$$

Cercignani Conjecture

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For the Kac model

$$-\sup_F \frac{\dot{S}(F|F_R)}{S(F|F_R)} \geq \frac{1}{M}$$

but for every δ there exists C_δ and F_δ such that

$$-\frac{\dot{S}(F_\delta|F_R)}{S(F_\delta|F_R)} \leq \frac{C_\delta}{M^{1-\delta}}.$$

Villani (2003), Einav (2011)
Mischler and Muhot obtained polynomial decay unifrom in N.

Given two probability distributions $F(V)$ and $G(V)$ on $\mathbb{R}^M$, symmetric in the V variables and with 0 average, that is

$$\int_{\mathbb{R}^M} F(V)dV = 1 \qquad \int_{\mathbb{R}^M} v_i F(V)dV = 0$$

and analogously for G, we can define the GTW distance as follows. Let

$$\hat{F}(\Theta) = \int_{\mathbb{R}^M} e^{i(V\cdot\Theta)} F(V)dV.$$

that is $\hat{F}(\Theta)$ is the Fourier transform of $F(V)$. Then we define

$$d_2(F,G) = \sup_{\Theta\neq 0} \frac{|\hat{F}(\Theta) - \hat{G}(\Theta)|}{\|\Theta\|^2}$$

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It is easy to see that this is a metric.

Convergence in the GTW metric.

It is quite easy to see that $\mathcal{L}_S$ is not expanding with respect to d_2 that is

$$d_2(e^{\mathcal{L}_S t}F, e^{\mathcal{L}_S t}G) \leq d_2(F, G)$$

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With more effort one can prove that (Tossounian, 2016):

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Moreover if

$$F(V) = \prod_{i=1}^{M} f(v_i) \qquad \text{and} \qquad G(V) = \prod_{i=1}^{M} g(v_i)$$

then

$$d_2(F, G) = d_2(f, g).$$

that is, the GTW metric is *extensive*.

Local Perturbation

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We write M as $N + M$ with $M << N$, the state of the system as

$$F_t(V, W) \qquad V \in \mathbb{R}^M \qquad W \in \mathbb{R}^N$$

and the generator as

$$\mathcal{L} = Q - I \qquad Q = \frac{1}{\binom{N+M}{2}} \sum_{1 \le i < j \le N+M} R_{i,j}$$

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that is we can write

$$Q = \frac{2}{N+M-1} \sum_{1 \le i < j \le M} R_{i,j} + \frac{2}{N+M-1} \sum_{M+1 \le i < j \le N} R_{i,j} + \frac{2}{N+M-1} \sum_{i=1}^{M} \sum_{j=M+1}^{N+M} R_{i,j} \, .$$

Finally we choose the initial conditions as

$$F_0(V,W) = f_0(V)e^{-\pi|W|^2}.$$

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Thus we look at the evolution of an initial state where M particles are out of equilibrium while the remaining N are in a canonical equilibrium at temperature $T = \frac{1}{2\pi}$.

Since we are mostly interested in the evolution of the M particles in the local “volume element” we can look at the marginal of F_t

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$$S(f_t|f_\infty) = \int f_t(V) \log\left(\frac{f_t(V)}{f_\infty(V)}\right) dV$$

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and again try to prove that

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This looks more promising but there are still problems.

It is not hard to see that

$$S\left(f_\infty \middle| e^{-\pi|V|^2}\right) = O\left(\frac{1}{N+M}\right)$$

thus we decide to look at

$$S(f_t) = S\left(f_t \middle| e^{-\pi|V|^2}\right).$$

that is we look at the entropy relative to the distribution in the Canonical Ensemble.

Theorem

Assume that the initial state of the system is of the form

$$F_0(V,W) = f_0(V)e^{-\pi|W|^2}.$$

with

$$S(f_0) = \int_{\mathbb{R}^M} f_0(V)\log\left(\frac{f_0(V)}{e^{-\pi|V|^2}}\right) dV < \infty$$

and define

$$f_t(V) = \int_{\mathbb{R}^N} F_t(V,W)\,dW = \int_{\mathbb{R}^N} \left(e^{\mathcal{L}t}F_0\right)(V,W)\,dW$$

then if $N > M$ we have

$$S(f_t) \leq \left(\frac{1}{N+M} + \frac{N}{N+M}e^{-\frac{1}{2}\frac{N+M}{N+M-1}t}\right) S(f_0)$$

The result is more general. We can write the generator as

$$Q = \frac{\lambda_M}{M-1} \sum_{1\le i<j\le M} R_{i,j} + \frac{\lambda_N}{N-1} \sum_{M+1\le i<j\le N} R_{i,j} + \frac{\mu}{N} \sum_{i=1}^{M} \sum_{j=M+1}^{N+M} R_{i,j}$$

and get

$$S(f_t) \le \left[\frac{M}{N+M} + \frac{N}{N+M} e^{-t\frac{\mu}{2}(N+M)/N} \right] S(f_0)$$

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independently from λ_M and λ_N.

The previous case correspond to

$$\lambda_M = \frac{2(M-1)}{N+M-1}\ ,\ \lambda_N = \frac{2(N-1)}{N+M-1}\ \text{ and }\ \mu = \frac{2N}{N+M-1}\ .$$

Taking λ_M and λ_N independent from N and M we can interpret the above system as a large reservoir with N particles in contact with a small system with M particles.

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In this case we can prove that, as $N \to \infty$, the combined evolution of system+reservoir converge uniformly in time to the evolution of a small Kac system with M particles interacting with a Maxwellian thermostat.

We can prove this convergence both in a suitable L^2 norm and in the GTW d_2 metric but we cannot get it in relative entropy.

The expansion

We start expanding the exponential as

$$F_t(V,W) = e^{-(M+N)t} \sum_{k=0}^{\infty} \frac{(M+N)^k t^k}{k!} Q^k F_0(V,W)$$

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we can further expand

$$Q^k F_0(V,W) = \lambda^k \sum_{\alpha_1,\ldots,\alpha_k}\int \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_k}{2\pi}F_0([\Pi_{j=1}^k r_{\alpha_j}(\theta_j)]^{-1}(V,W))$$

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where

$$\lambda = \frac{2}{(M+N)(M+N-1)}$$

and $\alpha = (i,j)$ indicates a pair of particles.

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and $\alpha = (i,j)$ indicates a pair of particles.

Thus we write the evolution as an average over all possible “collision histories”.

Write

$$f(V) = h(V)e^{-\pi|V|^2}$$

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so that

$$F_0(V, W) = (h \circ P)(V, W)e^{-\pi(|V|^2+|W|^2)}$$

where

$$P : \mathbb{R}^{M+N} \to \mathbb{R}^M \, , P(V, W) = V \, .$$

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We get

$$Q^k F_0(V, W) = e^{-\pi|V|^2}\lambda^k \sum_{\alpha_1,\ldots,\alpha_k} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi}(h \circ P)([\Pi_{l=1}^k r_{\alpha_l}(\theta_l)]^{-1}(V, W))e^{-\pi|W|^2}.$$

Integrating over W

$$\int_{\mathbb{R}^N} Q^k F_0(V,W)dW = e^{-\pi|V|^2}\lambda^k \sum_{\alpha_1,\ldots,\alpha_k} \int \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_k}{2\pi}\,\mathcal{N}_{k,\underline{\alpha},\underline{\theta}}h(W)$$

where $\underline{\alpha} = (\alpha_1,\ldots,\alpha_k)$, $\underline{\theta} = (\theta_1,\ldots,\theta_k)$ and

$$\mathcal{N}_{k,\underline{\alpha},\underline{\theta}}h(V) = \int_{\mathbb{R}^N} (h\circ P)([\Pi_{j=1}^k r_{\alpha_j}(\theta_j)]^{-1}(V,W))e^{-\pi|W|^2}dW$$

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Putting all together and using convexity of the entropy we get

$$S(f_t) \leq e^{-(M+N)t}\sum_{k=0}^{\infty}\frac{(M+N)^k t^k}{k!}\lambda^k \sum_{\alpha_1,\ldots,\alpha_k}\int\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_k}{2\pi}\times$$
$$\int_{\mathbb{R}^M} \mathcal{N}_{k,\underline{\alpha},\underline{\theta}} h(V)\log\left[\mathcal{N}_{k,\underline{\alpha},\underline{\theta}} h(V)\right] e^{-\pi|V|^2}dV$$

Call

$$\mathcal{S}(h) = \int_{\mathbb{R}^M} h(V) \log h(V) e^{-\pi |V|^2}\, dV$$

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then we need

$$\mathcal{S}(Q^k h) = \lambda^k \sum_{\alpha_1,\ldots,\alpha_k} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \mathcal{S}(\mathcal{N}_{k,\underline{\alpha},\underline{\theta}} h) \leq C_{k,M} \mathcal{S}(h)$$

where

$$\mathcal{N}_{k,\underline{\alpha},\underline{\theta}} h(V) = \int_{\mathbb{R}^N} (h \circ P)([\Pi_{j=1}^k r_{\alpha_j}(\theta_j)]^{-1}(V, W)) e^{-\pi |W|^2} dW$$

and

$$C_{k,M} = \left[\frac{M}{N+M} + \frac{N}{N+M} \left(1 - \frac{1}{2} \frac{N+M}{N+M-1} \right)^k \right]$$

Indeed we find

$$S(f_t) \leq e^{-(M+N)t} \sum_{k=0}^{\infty} \frac{(M+N)^k t^k}{k!} \lambda^k \sum_{\alpha_1,\ldots,\alpha_k} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \times$$

$$\int_{\mathbb{R}^M} \mathcal{N}_{k,\underline{\alpha},\underline{\theta}} h(V) \log \left[\mathcal{N}_{k,\underline{\alpha},\underline{\theta}} h(V)\right] e^{-\pi |V|^2} dV$$

$$\leq e^{-(M+N)t} \sum_{k=0}^{\infty} \frac{(M+N)^k t^k}{k!} \left[\frac{M}{N+M} + \frac{N}{N+M}\left(1 - \frac{1}{2}\frac{N+M}{N+M-1}\right)^k\right] S(f_0)$$

$$= \left[\frac{M}{N+M} + \frac{N}{N+M} e^{-\frac{t}{2}\frac{N+M}{N+M-1}}\right] S(f_0)$$

Write

$$O_k(\underline{\alpha},\underline{\theta}) = \left[\prod_{j=1}^{k} r_{\alpha_j}(\theta_j)\right]^{-1} = \begin{bmatrix} A_k(\underline{\alpha},\underline{\theta}) & B_k(\underline{\alpha},\underline{\theta}) \\ C_k(\underline{\alpha},\underline{\theta}) & D_k(\underline{\alpha},\underline{\theta}) \end{bmatrix}$$

$$A_k(\underline{\alpha},\underline{\theta})A_k(\underline{\alpha},\underline{\theta})^T + B_k(\underline{\alpha},\underline{\theta})B_k(\underline{\alpha},\underline{\theta})^T = I_M$$

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$$(h \circ P)([\Pi_{j=1}^{k} r_{\alpha_j}(\theta_j)]^{-1}(V, W)) = h(A_k(\underline{\alpha},\underline{\theta})V + B_k(\underline{\alpha},\underline{\theta})W)$$

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Partially integrating W

$$\begin{aligned}\mathcal{N}_{k,\underline{\alpha},\underline{\theta}}h(V) &= \int_{\mathbb{R}^N} h(A_k(\underline{\alpha},\underline{\theta})V + B_k(\underline{\alpha},\underline{\theta})W)e^{-\pi|W|^2}dW \\ &= \int_{\mathbb{R}^M} h\left(A_k(\underline{\alpha},\underline{\theta})V + \left(I_M - A_k(\underline{\alpha},\underline{\theta})A_k(\underline{\alpha},\underline{\theta})^T\right)^{1/2} W\right) e^{-\pi|W|^2}dW\end{aligned}$$

Ornstein-Uhlenbeck operator with matrix valued times

Nelson's hypercontractive estimate

Call

$$(\mathcal{N}_a h)(v) = \int_{\mathbb{R}} h(av + (1 - a^2)^{1/2} w) e^{-\pi w^2} dw$$

where

$$a^2 = e^{-t} \leq 1$$

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where

$$a^2 = e^{-t} \leq 1$$

Theorem

Assume that $h : \mathbb{R} \to \mathbb{R}_+$ has finite entropy, i.e.,

$$S(h) = \int_{\mathbb{R}} h(v) \log h(v) e^{-\pi v^2} dv < \infty$$

then

$$\mathcal{S}(\mathcal{N}_a h) \leq a^2 \mathcal{S}(h) + (1 - a^2) \|h\|_1 \log \|h\|_1$$

Write $A_k(\underline{\alpha}, \underline{\theta})$ as (SVD):

$$A_k(\underline{\alpha}, \underline{\theta}) = U_k(\underline{\alpha}, \underline{\theta})\Gamma_k(\underline{\alpha}, \underline{\theta})V_k(\underline{\alpha}, \underline{\theta})^T$$

where

$$\Gamma_k(\underline{\alpha}, \underline{\theta}) = \mathrm{diag}\left(\gamma_1, \cdots, \gamma_M\right)\ ,\ 0 \leq \gamma_j \leq 1$$

and $U_k(\underline{\alpha}, \underline{\theta})$, $V_k(\underline{\alpha}, \underline{\theta})$ are unitary.

Matrix valued time.

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and $U_k(\underline{\alpha}, \underline{\theta})$, $V_k(\underline{\alpha}, \underline{\theta})$ are unitary.

Theorem

Let $h \in L^1(\mathbb{R}^M, e^{-\pi|V|^2} dV)$ and assume that $\mathcal{S}(h) < \infty$. Then

$$\mathcal{S}(\mathcal{N}_{A_k(\underline{\alpha}, \underline{\theta})} h) \le \sum_{\sigma \subset \{1, \ldots, M\}} \Pi_{i \in \sigma^c} \gamma_i^2 \Pi_{j \in \sigma} (1 - \gamma_j^2) \mathcal{S}(h^\sigma_{U_k(\underline{\alpha}, \underline{\theta})})$$

where the σ marginal h^σ_U is given by

$$h^\sigma_U(Z) = \int_{\mathbb{R}^\sigma} h(U(Z', Z)) e^{-\pi|Z|^2} dZ'$$

Collecting everything we get

$$\mathcal{S}(Q^k h) \leq \sum_{\alpha_1,\ldots,\alpha_k} \lambda^k \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi}$$

$$\sum_{\sigma \subset \{1,\ldots,M\}} \Pi_{i\in\sigma^c} \gamma_{k,i}(\underline{\alpha},\underline{\theta})^2 \Pi_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha},\underline{\theta})^2)$$

$$\int_{\mathbb{R}^M} h(V) \log h^{\sigma}_{U_k(\underline{\alpha},\underline{\theta})}(P_{\sigma^c} U_k(\underline{\alpha},\underline{\theta})^T V) e^{-\pi|V|^2} dV\ .$$

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while we need

$$\mathcal{S}(Q^k h) \leq C_{k,M} \mathcal{S}(h)\,.$$

Collecting everything we get

$$\mathcal{S}(Q^k h) \leq \sum_{\alpha_1,\ldots,\alpha_k} \lambda^k \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi}$$
$$\sum_{\sigma\subset\{1,\ldots,M\}} \Pi_{i\in\sigma^c}\gamma_{k,i}(\underline{\alpha},\underline{\theta})^2 \Pi_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha},\underline{\theta})^2)$$
$$\int_{\mathbb{R}^M} h(V) \log h^{\sigma}_{U_k(\underline{\alpha},\underline{\theta})}(P_{\sigma^c} U_k(\underline{\alpha},\underline{\theta})^T V) e^{-\pi|V|^2} dV \,.$$

while we need

$$\mathcal{S}(Q^k h) \leq C_{k,M} \mathcal{S}(h) \,.$$

We will use the Brascamp-Lieb Inequality.

A simple case is:

Lemma

Let $h(V)$ be such that

$$\int_{\mathbb{R}^M} h(V)e^{-\pi|V|^2}dV = 1$$

and let its marginal over the j-th variable be denoted by

$$h_j(V^j) = \int h(V)e^{-\pi|V_j|^2}dV_j\,,$$

where $V^j = (V_1, ..., V_{j-1}, V_{j+1}, ..., V_N)$. Then we have

$$\sum_{j=1}^{N}\int h\log h_j\, e^{-\pi|V|^2}dV \le (N-1)\int h\log h\, e^{-\pi|V|^2}dV\,.$$

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$$\sum_{j=1}^N \int h\log h_j\, e^{-\pi|V|^2}dV \le (N-1)\int h\log h\, e^{-\pi|V|^2}dV\,.$$

The Lemma easily follows from the Loomis-Whitney inequality

$$\int_{\mathbb{R}^M} F_1(V^1)\cdots F_M(V^M)dV \le \|F_1\|_{L^{M-1}}\cdots\|F_M\|_{L^{M-1}}$$

where $F_i \in L^{M-1}(\mathbb{R}^{M-1})$.

Theorem

For $i = 1, \ldots K$, let

1. $H_i \subset \mathbb{R}^M$ *be subspaces of dimension* d_i*;*
2. $B_i : \mathbb{R}^M \to H_i$ *linear maps such that* $B_i B_i^T = I_{H_i}$*.*
3. $f_i : H_i \to \mathbb{R}$ *non nengative functions.*
4. c_i *non negative constants such that*

$$\sum_{i=1}^{K} c_i B_i^T B_i = I_M \ .$$

then for any non-negative function $h \in L^1(\mathbb{R}^M, e^{-\pi|V|^2} dV)$ *with* $\|h\|_1 = 1$ *we get*

$$\int_{\mathbb{R}^M} h(V) \log h(V) e^{-\pi|V|^2} dV \geq$$

$$\geq \sum_{i=1}^{K} c_i \int_{\mathbb{R}^M} \left[h(V) \log f_i(B_i V) e^{-\pi|V|^2} dV - \log \int_{H_i} f_i(u) e^{-\pi u^2} du \right]$$

Let:

$$
\begin{aligned}
f_i(u) &\longleftrightarrow h_U^\sigma(V) \\
H_i &\longleftrightarrow \mathbb{R}^{\sigma^c} \\
B_i &\longleftrightarrow P_{\sigma^c} U_k(\underline{\alpha}, \underline{\theta})^T \\
c_i &\longleftrightarrow \frac{\lambda^k}{C_{k,M}} \prod_{l=1}^{k} \frac{d\theta_l}{2\pi} \prod_{i\in\sigma^c} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \prod_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2)
\end{aligned}
$$

Let:

$$\begin{aligned} f_i(u) &\longleftrightarrow h_U^\sigma(V) \\ H_i &\longleftrightarrow \mathbb{R}^{\sigma^c} \\ B_i &\longleftrightarrow P_{\sigma^c} U_k(\underline{\alpha}, \underline{\theta})^T \\ c_i &\longleftrightarrow \frac{\lambda^k}{C_{k,M}} \prod_{l=1}^{k} \frac{d\theta_l}{2\pi} \prod_{i\in\sigma^c} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \prod_{j\in\sigma} (1 - \gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2) \end{aligned}$$

and assume that

$$\sum_{\alpha_1,\ldots,\alpha_k} \lambda^k \int \prod_{l=1}^{k} \frac{d\theta_l}{2\pi} \sum_{\sigma\subset\{1,\ldots,M\}} \prod_{i\in\sigma^c} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \prod_{j\in\sigma} (1 - \gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2) \times$$
$$U_k(\underline{\alpha}, \underline{\theta}) P_{\sigma^c}^T P_{\sigma^c} U_k(\underline{\alpha}, \underline{\theta})^T = C_{k,M} I_M \ .$$

where

$$C_{k,M} = \left[\frac{M}{N+M} + \frac{N}{N+M} \left(1 - \mu(\rho) \frac{N+M}{N+M-1} \right)^k \right]$$

Applying Brascamp-Lieb

Let:

$$f_i(u) \longleftrightarrow h_U^{\sigma}(V)$$

$$H_i \longleftrightarrow \mathbb{R}^{\sigma^c}$$

$$B_i \longleftrightarrow P_{\sigma^c} U_k(\underline{\alpha}, \underline{\theta})^T$$

$$c_i \longleftrightarrow \frac{\lambda^k}{C_{k,M}} \prod_{l=1}^{k} \frac{d\theta_l}{2\pi} \prod_{i\in\sigma^c} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \prod_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2)$$

and assume that

$$\sum_{\alpha_1,\ldots,\alpha_k} \lambda^k \int \prod_{l=1}^{k} \frac{d\theta_l}{2\pi} \sum_{\sigma\subset\{1,\ldots,M\}} \prod_{i\in\sigma^c} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \prod_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2) \times U_k(\underline{\alpha}, \underline{\theta}) P_{\sigma^c}^T P_{\sigma^c} U_k(\underline{\alpha}, \underline{\theta})^T = C_{k,M} I_M \ .$$

where

$$C_{k,M} = \left[\frac{M}{N+M} + \frac{N}{N+M}\left(1-\mu(\rho)\frac{N+M}{N+M-1}\right)^k\right]$$

then the Brascamp-Lieb inequality delivers exactly what we need.

Summation Formula

Since

$$U_k(\underline{\alpha},\underline{\theta})P^T_{\sigma^c}P_{\sigma^c}U_k(\underline{\alpha},\underline{\theta})^T = I_M$$

summing over σ, we get that we need to show

$$\sum_{\alpha_1,\dots,\alpha_k} \lambda^k \int \prod_{l=1}^{k} \frac{d\theta_l}{2\pi} A_k(\underline{\alpha},\underline{\theta})^T A_k(\underline{\alpha},\underline{\theta}) = C_{k,M}\, I_M\ .$$

Since

$$U_k(\underline{\alpha},\underline{\theta})P_{\sigma^c}^T P_{\sigma^c} U_k(\underline{\alpha},\underline{\theta})^T = I_M$$

summing over σ, we get that we need to show

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Remember

$$O_k(\underline{\alpha},\underline{\theta}) = \left[\prod_{j=1}^{k} r_{\alpha_j}(\theta_j)\right]^{-1} = \begin{bmatrix} A_k(\underline{\alpha},\underline{\theta}) & B_k(\underline{\alpha},\underline{\theta}) \\ C_k(\underline{\alpha},\underline{\theta}) & D_k(\underline{\alpha},\underline{\theta}) \end{bmatrix}$$

thus

$$A_k(\underline{\alpha},\underline{\theta})^T A_k(\underline{\alpha},\underline{\theta}) = O_k(\underline{\alpha},\underline{\theta})^T\, J\, O_k(\underline{\alpha},\underline{\theta})\Big|_{M\times M} \qquad J = \begin{pmatrix} I_M & 0 \\ 0 & 0 \end{pmatrix}$$

Let

$$J(\underline{m}) = \begin{pmatrix} m_1 I_M & 0 \\ 0 & m_2 I_N \end{pmatrix}$$

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we get

$$\sum_{\alpha} \lambda \int \frac{d\theta}{2\pi} \, r_\alpha(\theta) \, J(\underline{m}) \, r_\alpha(\theta)^{-1} = J(\underline{m}')$$

where

$$\underline{m}' = \mathcal{P}\underline{m} \qquad\qquad \mathcal{P} = I_2 + \frac{\mu(\rho)}{N+M-1} \begin{pmatrix} N & -N \\ -M & M \end{pmatrix}$$

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$$\underline{m}' = \mathcal{P}\underline{m} \qquad\qquad \mathcal{P} = I_2 + \frac{\mu(\rho)}{N+M-1} \begin{pmatrix} N & -N \\ -M & M \end{pmatrix}$$

so that

$$\sum_{\alpha_1,\ldots,\alpha_k} \lambda^k \int \prod_{l=1}^{k} \rho(\theta_l) d\theta_l \, A_k(\underline{\alpha},\underline{\theta})^T A_k(\underline{\alpha},\underline{\theta}) = \left(\mathcal{P}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)_1 I_M$$

That is exactly what we needed.

Thank You.